On the large subgroups of $O_1(n)$ and a result by Kowalsky and Witte.

Vic Patrangenaru *
Department of Statistics
Florida State University, Tallahassee, FL 32304, U.S.A.
vic@stat.fsu.edu

Abstract

One shows that the maximal Lie subgoups of the Lorentz group $O_1(n)$ of first and second largest possible dimensions are the groups of Lorentz transformations that leave invariant a vector hyperplane in the Minkowski space. One also shows that the only Lorentz manifolds that admit an almost simple transitive group of isometries with a noncompact stabilizer are the de-Sitter and the anti de-Sitter spaces.

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1 Introduction.

It is known that the maximal Lie subroups of largest dimension of $SO(n)$ are conjugated with $SO(n-1)$ [5]. If we consider that natural representation of $SO(n)$ on the Euclidean space $\mathbb{E}^n$, one may see that a maximal Lie subroup of largest dimension of $SO(n)$ is the Lie subgroup of all transformations in $SO(n)$ that keep a vector line in $\mathbb{E}^n$ fixed.
In this note, we determine the maximal Lie subroups of first two largest dimensions of the Lorentz group $O_1(n)$. The answer is complicated by the fact that $O_1(n)$ is not compact, and some of its subgroups leave invariant a vector subspace S of $\mathbb{R}^n$ where the Minkowski squared "norm" $\| \ \|_1^2$ is degenerated, so that $S \cap S^{\perp_1} \neq 0$. To investigate this question we look at the Lie algebra level and get the information we need using an elementary approach. A first result in this direction [8] claims that the for $n > 2, n \neq 5$, the largest possible dimension of a Lie subalgebra $\mathfrak{a}$ of $\mathfrak{o}_1(n)$ is $\frac{1}{2}n(n-1)+2$ and any $\mathfrak{a}$ is conjugate to the Lie agebra of the group of transformations that leave invariant an isotropic vector hyperplane of $\mathbb{E}_1^n$.
In Theorem 2.1 we describe the $\frac{1}{2}n(n-1)+1$-dimensional Lie subalgebras $\mathfrak{a}$ of $\mathfrak{o}_1(n)$, and in Corollary 2.1 we identify the corresponding maximal Lie subgroups of $O_1(n)$.

In section 3, we determine the homogeneous pseudo-Riemannian structures on the quotient spaces $O_1(m+1)/H$, where $m = n-1$ and H is one of the maximal subroups in section 2. We then shed a more geometric light on a recent result of D. Witte [11].

2 $\frac{1}{2}n(n-1)+1$-dimensional Lie subalgebras of the Lorentz algebra

We specify our notations first. If $x = (x^1, ..., x^n) \in \mathbb{R}^n$, then $\|x\|_1^2 = (x^1)^2 + ... + (x^{n-1})^2 - (x^n)^2 = \sum_{i=1,j=1}^{n} {}_1\delta_{ij} x^i x^j$, where

$$
{}_1\delta_{ij} = \begin{cases} 0 & , \quad i \neq j \\ 1 & , \quad i = j < n \\ -1 & , \quad i = j = n \end{cases}
$$

The Minkowski space is $\mathbb{E}_1^n = (\mathbb{R}^n, \| \quad \|_1^2)$. The standard orthobasis in $\mathbb{E}_1^n$ is denoted by $e_i, i = 1..., n$. The Lorentz group $O_1(n)$ is the group of automorphisms of $\mathbb{E}_1^n$; if L_n is the matrix

$$
\begin{pmatrix} I_{n-1} & 0 \\ 0 & I_1 \end{pmatrix},
$$

then $Q \in O_1(n) \Leftrightarrow Q^t L_n Q = L_n$. The Lie algebra of this group, the Lorentz algebra is $\mathfrak{o}_1(n)$. Let us denote by e_i^j the matrix , defined by $e_i^j e_k = \delta_k^j e_i, \forall k = 1, ..., n$, and consider the matrices

$$
f_i^j = e_i^j -_1 \delta_{ij} e_j^i, \forall i, j \leq n
$$

Note that $f_i^j +_1 \delta_{ij} f_j^i = 0$ and $(f_i^j)_{i<j\leq n}$ forms a basis of $\mathfrak{o}_1(n)$. The muliplication rule follows from the standard multiplication is $\mathfrak{gl}(n)$. The only nonzero products are

$$
[f_i^j, f_j^k] = f_i^k, \forall i < n, \forall j, k \leq n, i \neq j \neq k. \tag{1}
$$

In [8] it was shown that the first gap in dimensions of Lie subalgebras of $\mathfrak{o}_1(n)$ is given by the interval $(\frac{1}{2}(n-1)(n-2)+1, \frac{1}{2}(n-1)n)$ and any

Lie subalgebra of dimension $\frac{1}{2}(n-1)(n-2)+1$ is conjugated with

$$\mathfrak{m}(n) = \mathfrak{so}(n-1) \oplus \mathbb{R}f_{n-1}^{n} \oplus Span(\{f_i^{n-1} + f_i^n, i = 1, ..., n-2\})$$

Let $\mathfrak{h}(n)$ be defined by

$$\mathfrak{h}(n) =: \mathfrak{so}(n-1) \oplus Span(\{f_i^{n-1} + f_i^n, i = 1, ..., n-2\})$$

Theorem 2.1 *Assume $n > 5$. A $\frac{1}{2}(n-1)(n-2)$-dimensional Lie subalgebra of $\mathfrak{o}_1(n)$ is conjugated with one and only one of the following subalgebras:* $\mathfrak{o}(n-1), \mathfrak{o}_1(n-1), \mathfrak{h}(n)$.

PROOF. Let $\mathfrak{h}$ be a $d = \frac{1}{2}(n-1)(n-2)$-dimensional Lie subalgebra of $\mathfrak{o}_1(n)$ The orthogonal algebra $\mathfrak{o}(n-1)$ can be regarded as a Lie subalgebra of $\mathfrak{o}_1(n)$, by identifying $T \in \mathfrak{o}(n-1)$ with the matrix

$$\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix},$$

Let $\mathfrak{a} =: \mathfrak{h} \cap \mathfrak{o}(n-1)$, and consider the map :

$$j : \mathfrak{o}(n-1)/\mathfrak{a} \to \mathfrak{o}_1(n)/\mathfrak{h}, \quad j(\xi mod\mathfrak{a}) = \xi mod\mathfrak{h}$$

It is obvoius that j is a monomorphism of Lie algebras, and therefore

$$dim\mathfrak{a} \geq dim\mathfrak{o}(n-1) - dim\mathfrak{o}_1(n) + dim\mathfrak{h} = \frac{1}{2}(n-3)(n-2) - 1 \quad (2)$$

Assume $\mathfrak{a}$ is a proper Lie subalgebra of $\mathfrak{o}(n-1)$. Since $n > 5$, any Lie subalgebra of $\mathfrak{o}(n-1)$ of dimension at least $\frac{1}{2}(n-3)(n-2)-1$ is conjugated with $\mathfrak{o}(n-2)$ and w.l.o.g. we may assume that $\mathfrak{h} = \mathfrak{o}(n-2) \oplus \mathfrak{p}$, where $\mathfrak{p}$ is included in the orthocomplement $\mathfrak{b}$ of $\mathfrak{o}(n-2)$ in $\mathfrak{o}_1(n)$ w.r.t. the Kiling form.

For $j = 1, ..., n-2$, let $\mathfrak{b}_j = \mathbb{R}f_j^{n-1} \oplus \mathbb{R}f_j^n$ and $\mathfrak{b}_{n-1} = \mathbb{R}f_{n-1}^n$. For any pair $i \neq j \leq n-2, adf_i^j(\mathfrak{b}_i) = \mathfrak{b}_j$ and $adf_i^j(\mathfrak{p}) \subseteq \mathfrak{p}$. Therefore the subspaces $\mathfrak{b}_i \cap \mathfrak{p}, i = 1, ..., n-2$ are all isomorphic. Moreover they are unidimensional, since $dim\mathfrak{p} = dim\mathfrak{h} - dim\mathfrak{o}(n-2) = n-2$ and in addition $\mathfrak{p} \cap \mathfrak{b}_{n-1} = 0$.

If $i, j \leq n-2$ and $bf_j^{n-1} + f_j^n$ is a generator of $\mathfrak{b}_j$, then $bf_i^{n-1} + f_i^n = adf_i^j(bf_j^{n-1} + f_j^n)$ is a generator of $\mathfrak{b}_i$.

Let us consider the hyperbolic rotation $R(t)$ given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & cosh(t) & sinh(t) \\ 0 & sinh(t) & cosh(t) \end{pmatrix},$$

If $|b| < 1$, a direct computation shows that if we select $tanh(t) = b$ then $Ad(R(t))(\mathfrak{h}) = \mathfrak{o}_1(n-1)$.

If $|b| > 1$, a similar computation shows that if we select $coth(t) = b$ then $Ad(R(t))(\mathfrak{h}) = \mathfrak{o}(n-1)$.

If $b = 1$, then $\mathfrak{h} = \mathfrak{h}(n)$. Finally if $b = -1$, one can easily show that $\mathfrak{h}$ is conjugated with $\mathfrak{h}(n)$■

In [8] it was shown that any three maximal Lie subalgebra of $\mathfrak{o}_1(4)$ is conjugated with the Lie subalgebra of the Lorentz subgroup of $O_1(4)$, leaving invariant a hyperplane in the Minkowski space $\mathbb{E}_1^4$. Here we show that a similar result holds in higher dimensions.

Corollary 2.2 *Assume $n > 5$. Then a Lie subalgebra $\mathfrak{m}$ of the Lorentz algebra $\mathfrak{o}_1(n)$ of dimension $\frac{1}{2}(n-1)(n-2)$ or higher is maximal, iff $\mathfrak{m}$ is the Lie algebra of the group of Lorentz transformations leaving invariant* *a hyperplane in the Minkowski space* $\mathbb{E}^n_1$.

PROOF. The result follows from Theorem 2.1, and the observation that $\mathfrak{h}(n)$ is a Lie subalgebra of $\mathfrak{m}(n)$; in [8] it was shown that $\mathfrak{m}(n)$ is a maximal subalgebra of $\mathfrak{o}_1(n)$ ■

3 On a result by Kowalsky and Witte

The results in Section 2 were needed to classify the Lorentz manifolds in the first three strata [7], a classification essentially based on the $\mathfrak{h}$-triple method [9] of recovering homogeneous pseudo-Riemannian structures, from their isotropic subalgebras, which has its roots in the Cartan procedure of adapted frames on Riemannian homogeneous spaces [1], [10], [9].

In this section we will determine the homogeneous pseudo-Riemannian structures on the quotients spaces $O_1(m+1)^o/H$, where $m = n-1$ and H is the connected subgroup of $O_1(m+1)^o$ of Lie algebra $\mathfrak{h}$, where $\mathfrak{h}$ is one of the Lie subalgebras in section 2.

N. Kowalsky ([4], Thm. 5.1) showed that if K is an almost simple Lie group with finite center, that acts as a transitive group of isometries of a Lorentz manifold (M, g), and if K does not act as a transitive group of isometries of (M, g'), for any Riemannian structure g' on M, then K is locally isomorphic to $O_1(m+1)$ or to $O_2(m+1)$. As an immediate corollary, it follows that if K is an almost simple Lie group with finite center, that acts as a transitive group of isometries of a Lorentz manifold (M, g) with a noncompact isotropy group, then K is locally isomorphic to $O_1(m+1)$

or to $O_2(m+1)$. Witte [11] determined recently the local structure of the isotropy Lie subgroup H of such a transitive action of $O_1(m+1)$ or of $O_2(m+1)$. In [11] one shows that if such an H is such an isotropy subgroup of a transitive action by isometries of $O_1(m+1)$ or of$O_2(m+1)$ on a Lorentz manifold, then the identity component H^o of H is conjugate to $O_1(m)^o$. It is natural then to determine all the invariant Lorentz metrics on $O_1(m+1)^o/O_1(m)^o$ and on $O_2(m+1)^o/O_1(m)^o$.

Let M denote one of these manifolds and K be one of the corresponding groups of isometries of M, $K = O_1(m+1)^o$ or $K = O_2(m+1)^o$. Assume S_1, S_2 are space-like tangent planes at two arbitrary points x, y. Since M is homogeneous, there is an isometry f of M with $f(x) = y$. Let $S = d_x f(S_1)$. Since the isotropic representation of $O_1(n)^o$ is an open subgroup of $O_1(m)$, it follows that given the space-like tangent planes at $y \in M$, S, S_2, there is an $h \in O_1(m)^o$, with $d_y h(S) = S_2$; therefore $d_x(h \circ f(S_1) = S_2$, which shows that K is transitive on the set of all space-like planes tangent to M. From ([2], Theorem 2.a) it follows that M has constant curvature. With the notation in [6], we have $(O_1(m+1)^o/O_1(m)^o, g)$ is isometric to $S_1(m)$, the *de Sitter space* and $(O_2(m+1)^o/O_1(m)^o, g)$ is isometric to $H_1(m)$ the *anti de Sitter space*. Thus Witte's result can be formulated as follows:

Theorem 3.1 *(Kowalsky-Witte). The only Lorentz manifolds that admit a transitive group of K of isometries that is an almost simple Lie group with finite center, acting with a noncompact isotropy group, are the de-Sitter space and the anti de-Sitter space.*

The de Sitter space and the anti de Sitter space are spaces in the zero Lorentz stratum. To determine the structure of Lorentz manifolds in deeper Lorentz strata, a different technique is needed [7].

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