Project 2: Importance Sampling Using Tilted Distributions

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January 3, 2009
## Contents

1 Introduction ................................................. 2  
  1.1 Importance Sampling Estimator ......................... 2  
    1.1.1 Problem Statement ................................. 2  
2 Methodology .................................................. 3  
  2.1 Importance Sampling: The Importance Sampling Estimator . . 3  
  2.2 Determine the form of the tilted density .................. 3  
  2.3 From the tilted density, how to obtain samples ............... 4  
  2.4 State the optimal amount of tilt $t$ to estimate $\theta$, for a given $a$ . . 4  
  2.5 State the expression for $\tilde{\theta}_n$ Monte Carlo estimator that uses these samples. ................. 4  
  2.6 Compare the $\tilde{\theta}_n$ with $\hat{\theta}_n$, the classical Monte Carlo estimator that uses samples from $f(x)$ directly. .................. 5  
3 Matlab Code .................................................. 6  
4 Results ....................................................... 8  
  4.1 Comparing Classical MC and Important-Sampling ............... 8  
    4.1.1 Plots of Convergence ............................... 8  
    4.1.2 Computation Times .................................. 8  
    4.1.3 Accuracy and Consistency ........................... 9  
    4.1.4 Conclusion ........................................... 9  
  4.2 Comparison for different values of $a$ and $\lambda = 1$ ........ 10  
    4.2.1 $a = 2$ and $\lambda = 1$ ............................ 10  
    4.2.2 $a = 4$ and $\lambda = 1$ ............................ 11  
    4.2.3 $a = 6$ and $\lambda = 1$ ............................ 12  
    4.2.4 $a = 8$ and $\lambda = 1$ ............................ 13  
    4.2.5 Table of Estimates ................................. 14  
5 Conclusion ..................................................... 15
List of Figures

4.1 Convergence of Classical MC for \( a=6 \) and \( \lambda = 1 \) . . . . . . . . . 8
4.2 Convergence of Tilted Density for \( a=6 \) \( \lambda = 1 \) . . . . . . . . . . . 9
4.3 Convergence of Classical MC for \( a=2 \) and \( \lambda = 1 \) . . . . . . . . . 10
4.4 Convergence of Tilted Density for \( a=2 \) \( \lambda = 1 \) . . . . . . . . . . 10
4.5 Convergence of Classical MC for \( a=4 \) and \( \lambda = 1 \) . . . . . . . . . 11
4.6 Convergence of Tilted Density for \( a=4 \) \( \lambda = 1 \) . . . . . . . . . . 11
4.7 Convergence of Classical MC for \( a=8 \) and \( \lambda = 1 \) . . . . . . . . . 12
4.8 Convergence of Tilted Density for \( a=8 \) \( \lambda = 1 \) . . . . . . . . . . 12
4.9 Convergence of Classical MC for \( a=8 \) and \( \lambda = 1 \) . . . . . . . . . 13
4.10 Convergence of Tilted Density for \( a=8 \) \( \lambda = 1 \) . . . . . . . . . . 13
Chapter 1

Introduction

1.1 Importance Sampling Estimator

1.1.1 Problem Statement

The objective of this assignment is to estimate the right tail of an exponential random variable using ideas from importance sampling and tilted distributions.

The problem to solve for this project is as follows: Let $X$ be an exponential random variable with intensity $\lambda$ (mean $\frac{1}{\lambda}$). For an constant $a > 0$, the goal is to estimate the probability:

$$\theta = \Pr\{X > a\} = \lambda \int_{a}^{\infty} e^{-\lambda x} dx \quad (1.1)$$

Since the event $X > a$ occurs with a very small probability, i.e. 0 is very small, the use of classical Monte Carlo approach will not be efficient. The usual approach would be completely inadequate since approximating $\mu$ to any reasonable degree of accuracy would require $n$ to be very large. Using a much smaller value of $n$, would lead result in an estimate, $\hat{\theta} = 0$. Instead, it is better to use importance sampling with a tilted density serving as the density to sample from. In this paper, a method to reduce the variance of the estimator will be presented and numerical calculations of accuracy and computation time will be compared with classical approaches.
Chapter 2

Methodology

2.1 Importance Sampling: The Importance Sampling Estimator

To estimate $\theta = E[h(X)]$, where $X$ has a probability density function of $f(x)$. Let $g(\ldots)$ be another pdf with the property that $g(x) \neq 0$ whenever $f(x) \neq 0$. That is $g$ has the same support as $f$. Then

$$\theta = E[h(x)] = \int h(x)f(x)dx$$

$$= \int h(x)\frac{f(x)}{g(x)}g(x)dx \quad (2.1)$$

Since $g$ is a pdf, then the expectation with respect to $g(\ldots)$ is

$$\theta = E\left[\frac{h(X)f(X)}{g(X)}\right] \quad (2.2)$$

Using Monte Carlo simulation method generates $n$ samples of $X$ from the density, $f(\ldots)$, and set $\theta_n = \sum h(X)/n$. An alternative method is to generate $n$ values of $X$ from the density, $g(\ldots)$, and set

$$\hat{\theta}_{n,is} = \sum_{j=1}^{n} \frac{h(X_j)f(X_j)}{ng(X_j)} \quad (2.3)$$

then $\hat{\theta}_{n,is}$ is the importance sampling estimator of $\theta$.

2.2 Determine the form of the tilted density

For a scalar $t > 0$, define the tilted density as:

$$f_t(x) = \frac{e^{tx}f(x)}{M(t)}, \text{ where } f(x) = \lambda e^{\lambda x} \quad (2.4)$$
Then

\[ M'(x) = \int_0^\infty e^{tx} f(x)dx \]
\[ = \lambda \int_0^\infty e^{tx} e^{\lambda/x}dx \]
\[ = e^{-(\lambda-t)/x}dx \]
\[ = -\left(\frac{\lambda}{\lambda-t}\right)(0-1) \]
\[ = \left(\frac{\lambda}{\lambda-t}\right) \]

Therefore,

\[ f_t(x) = \frac{e^{tx} f(x)}{\left(\frac{\lambda}{\lambda-t}\right)} \]
\[ = (\lambda-t)e^{-(\lambda-t)x} \]

(2.5)

\[ f_t(x), \text{ the tilted density, has an exponential distribution with mean } = \frac{1}{(\lambda-t)}, \]
or using the reciprocal form, the intensity of \((\lambda-t)\).

2.3 From the tilted density, how to obtain samples

Will obtain samples by using the inverse transform method. Generate uniform random variables from [0,1] and \(X_i\)'s from \(\frac{1}{(\lambda-t)} \log(U)\).

2.4 State the optimal amount of tilt \(t\) to estimate \(\theta\), for a given \(a\).

From the mean of \(f_t(x)\) can obtain the \(t^*\), the optimal amount of tilt. The mean of \(f_t(x)\) is \(\frac{1}{(\lambda-t)}\), which is equal to \(a\). Therefore,

\[ t^* = \lambda - \left(\frac{1}{a}\right) \]

(2.7)

2.5 State the expression for \(\hat{\theta}_n\) Monte Carlo estimator that uses these samples.

By sampling \(X_i\)'s be from the optimally tilted density, say \(f^*_t(x)\), the \(X_i\)'s will have an exponential distribution with mean \(\frac{1}{(\lambda-t)}\).

From there, will substitute \(t^*\). This will make the \(X_i\)'s have exponential distribution with mean \(a\). Using the method described above, will generate the \(X_i\)'s to have a density of \(a \log(U)\).
Therefore the expression for 

\[
\tilde{\theta}_n = \frac{\lambda}{n(\lambda-t^*)} \sum_{i=1}^{n} I_{[a,\infty)}(x_i)e^{-t^* x_i}
\]

\[
= \frac{\alpha}{n} \sum_{i=1}^{n} I_{[a,\infty)}(x_i)e^{-(\lambda - \frac{1}{\lambda})x_i}
\]  

(2.8)

2.6 Compare the \(\tilde{\theta}_n\) with \(\hat{\theta}_n\), the classical Monte Carlo estimator that uses samples from \(f(x)\) directly.

From \(\hat{\theta}_n\), the classical Monte Carlo estimator, the \(X_i\)'s have an exponential distribution with mean \(\frac{1}{\lambda}\)

\[
\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^{n} I_{[a,\infty)}(x_i)
\]  

(2.9)
Chapter 3

Matlab Code

clear all; clc;
a=8;
lambda = 1;n=10000;
t=lambda -(1/a);
mt= lambda/(lambda-t);
u=rand(1,n);
x1=-log(u);
x = -a*log(u);
% classical estimate
 tic
 for i=1:length(x1)
     flag=0;
     if x1(i)>=a
         flag=flag+1;
     end
     if flag==1
         gx1(i)=1;
     else
         gx1(i)= 0;
     end
     thetahat(i)=(1/i)*sum(gx1);
 end
 toc
 figure(1)
 plot(thetahat)
 xlabel('n');
ylabel('thetahat')

% tilted sampling
 tic
 for i=1:length(x)
     flag=0;
     if x(i)>=a
         flag=flag+1;
     end
     if flag==1
         gx(i)=1;
     else
         gx(i)= 0;
     end
     thetahat(i)=(1/i)*sum(gx);
 end
 toc
 figure(1)
 plot(thetahat)
 xlabel('n');
ylabel('thetahat')
end
if flag==1
    gx(i)=exp(-t*x(i));
else
    gx(i)= 0;
end
thetatilda(i)=(mt/i)*sum(gx);
end
toc;
figure(2)
plot(thetatilda)
xlabel('n');
ylabel('thetatilda')

thetahat(i)
thetatilda(i)
Chapter 4

Results

4.1 Comparing Classical MC and Important-Sampling

4.1.1 Plots of Convergence

Implement the classical and the important-sampling estimators in Matlab. Compare the two estimators on the basis of: (a) computation time and (b) accuracy of estimation. In other words, take a sample size, say n = 10000, and study the computation time and the accuracy of the two estimators for that value of n. Choose \( a = 6 \) and \( \lambda = 1 \) for this study.

![Figure 4.1: Convergence of Classical MC for a=6 and \( \lambda = 1 \)](image)


4.1.2 Computation Times

<table>
<thead>
<tr>
<th>Estimator</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classical MC</td>
<td>0.566971 s</td>
</tr>
<tr>
<td>Important-sampling of tilted</td>
<td>0.530406 s</td>
</tr>
</tbody>
</table>

Figure 4.1: Convergence of Classical MC for a=6 and \( \lambda = 1 \)
4.1.3 Accuracy and Consistency

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>True Estimates</td>
<td>0.0025</td>
</tr>
<tr>
<td>Classical MC</td>
<td>0.0033</td>
</tr>
<tr>
<td>Important-sampling of tilted</td>
<td>0.0025</td>
</tr>
</tbody>
</table>

4.1.4 Conclusion

Analyzing the graphs, the classical MC approach took longer to converge. Also, in comparison with computation time, the important-sampling tilted was faster than the classical estimator. In accuracy the important-sampling tilted estimate was more accurate (closer to true value) and more consistent than the classical MC approach.
4.2 Comparation for different values of $a$ and $\lambda = 1$

Plot the convergence of $\hat{\theta}_n$ and $\tilde{\theta}_n$ versus $n$ for the following values of $a = 2, 4, 6,\text{and } 8$, with $\lambda = 1$. Plot them in the separate plots and clearly label your plots. Show your final estimates for these cases in a table.

4.2.1 $a = 2$ and $\lambda = 1$

![Figure 4.3: Convergence of Classical MC for $a=2$ and $\lambda = 1$](image1)

![Figure 4.4: Convergence of Tilted Density for $a=2 \lambda = 1$](image2)
4.2.2 $a = 4$ and $\lambda = 1$

Figure 4.5: Convergence of Classical MC for $a=4$ and $\lambda = 1$

Figure 4.6: Convergence of Tilted Density for $a=4$ $\lambda = 1$
4.2.3 \( a = 6 \) and \( \lambda = 1 \)

Figure 4.7: Convergence of Classical MC for \( a=6 \) and \( \lambda = 1 \)

Figure 4.8: Convergence of Tilted Density for \( a=6 \) \( \lambda = 1 \)
4.2.4  \( a = 8 \) and \( \lambda = 1 \)

Figure 4.9: Convergence of Classical MC for \( a=8 \) and \( \lambda = 1 \)

Figure 4.10: Convergence of Tilted Density for \( a=8 \) \( \lambda = 1 \)
### 4.2.5 Table of Estimates

<table>
<thead>
<tr>
<th>a</th>
<th>$\hat{\theta}_n$</th>
<th>$\tilde{\theta}_n$</th>
<th>True value</th>
<th>Comutational time $\hat{\theta}_n$</th>
<th>Comutational time $\tilde{\theta}_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.1377</td>
<td>0.1328</td>
<td>0.1353</td>
<td>0.568419 seconds.</td>
<td>0.530554 seconds.</td>
</tr>
<tr>
<td>4</td>
<td>0.0181</td>
<td>0.0185</td>
<td>0.0183</td>
<td>0.478063 seconds.</td>
<td>0.467150 seconds.</td>
</tr>
<tr>
<td>6</td>
<td>0.0024</td>
<td>0.0025</td>
<td>0.0025</td>
<td>0.566394 seconds.</td>
<td>0.443624 seconds.</td>
</tr>
<tr>
<td>8</td>
<td>4.0000e-004</td>
<td>3.4270e-004</td>
<td>3.34e-004</td>
<td>0.553530 seconds.</td>
<td>0.499536 seconds.</td>
</tr>
</tbody>
</table>
Chapter 5

Conclusion

In this paper, it was found that the tilted density estimator for $\theta$ calculated using importance sampling techniques was closer to the true value calculated from the integral. In comparison with the classical approach, the Monte Carlo approach, it was pointed in the tables and plots in the paper that the tilted density had shorter computational time, the accuracy was more precise, and convergence was faster and more stable. For rare events or events with small probability, such as the tail of the exponential density, the method implemented here proved to be a better estimator.