Elastic Geodesic Paths in Shape Space of Parametrized Surfaces

Sebastian Kurtek‡, Eric Klassen‡, John C. Gore∗, Zhaohua Ding∗, and Anuj Srivastava†

Abstract—This paper presents a novel Riemannian framework for shape analysis of parameterized surfaces. In particular, it provides efficient algorithms for computing geodesic paths which, in turn, are important for comparing, matching, and deforming surfaces. The novelty of this framework is that geodesics are invariant to the parameterizations of surfaces and other shape-preserving transformations of surfaces. The basic idea is to formulate a space of embedded surfaces (surfaces seen as embeddings of a unit sphere in $\mathbb{R}^3$) and impose a Riemannian metric on it in such a way that the re-parameterization group acts on this space by isometries. Under this framework, we solve two optimization problems. One, given any two surfaces at arbitrary rotations and parameterizations, we use a path-straightening approach to find a geodesic path between them under the chosen metric. Second, by modifying a technique presented in [24], we solve for the optimal rotation and parameterization (registration) between surfaces. Their combined solution provides an efficient mechanism for computing geodesic paths in shape spaces of parameterized surfaces. We illustrate these ideas using examples from shape analysis of anatomical structures and other general surfaces.

Index Terms—shape analysis, Riemannian distance, parameterization invariance, path-straightening, geodesics

1 INTRODUCTION

Shape is an important feature of objects and can be immensely useful in characterizing objects for the purpose of detection, tracking, classification, and recognition. As an example, it plays an important role in medical image analysis where advances in non-invasive imaging technology have enabled researchers to study biological variations of anatomical structures. Studying shapes of 3D anatomical structures in the brain is of particular interest because many diseases can potentially be linked to alterations of these shapes. Shape analysis of surfaces has also become important in biometrics, graphics, 3D TV, computer vision, etc.

There has been a significant amount of research and activity in the general area of shape analysis. By shape analysis we mean a set of tools for comparing, matching, deforming, and modelling shapes. The main differences amongst different tools proposed so far lie in the mathematical representations and metrics used in the analysis. For example, in shape analysis of planar objects (objects in 2D images), a variety of mathematical representations, including binary images, sampled points (active shape models [6]), ordered points (landmark-based shape analysis [10]), medial axes [31], level sets [29], and others, have been used. These different representations, along with their corresponding choices of metrics, lead to different solutions with their respective strengths and limitations. A natural representation for shape analysis of boundaries of planar objects is parametrized curves, although historically that representation has been underutilized. One of the main reasons for its limited use has been the issue of parameterization. While a re-parameterization of a curve does not change its shape, it does change the coordinate, angle, or curvature functions (as functions of the parameterization) along the curves and any comparison directly involving those functions will be affected. How can we deal with this shape-preserving but unconventional transformation? The solution comes from choosing representations and metrics in such a way that the resulting geodesics between curves are invariant to their re-parameterizations, in addition to the standard shape-preserving transformations such as rotation, translation and uniform scaling (see e.g. [37], [19], [38], [32]). For instance, Srivastava et al. [32], [18], [19] use a square-root velocity function (SRVF) $q(t) = \frac{\beta(t)}{\sqrt{\beta'(t)}},$ to analyze the shape of a parameterized curve $\beta : [0,1] \rightarrow \mathbb{R}^n$ under the standard $L^2$ metric. There are several reasons for selecting such a representation; two important ones are:

1) Under the $L^2$ metric, the action of the re-parameterization group is by isometries, i.e. if $q_1$, $q_2$ are two SRVF$s$ and $\gamma : [0,1] \rightarrow [0,1]$ is any re-parameterization function, then $\|q_1 - q_2\| = \|(q_1, \gamma) - (q_2, \gamma)\|,$ where $(q_1, \gamma)$ denotes the SRVF of the re-parameterized curve. This helps define a proper metric on the shape space (representation space modulo the re-parameterization group), and ultimately makes shape analysis invariant to re-parameterization using distances of the type: $\min_{\gamma} \|q_1 - (q_2, \gamma)\|.$

2) Under the SRVF representation an elastic Riemannian metric defined in [28] becomes the standard
L² metric enabling its use in the previous item.
3) This framework allows pair-wise matching of points on curves using optimal reparameterizations while simultaneously computing distances.

In this paper we are focused on shape analysis of boundaries formed by 3D objects. In particular, we focus on shape analysis of parametrized surfaces of genus zero, and we are interested in a Riemannian framework that allows comparison, matching, deformation, averaging, and modeling of observed shapes. Motivated by the Riemannian shape analysis of curves, we pose the following question: What is a natural representation of surfaces and a corresponding metric that together allow for a parameterization-invariant shape analysis? The solution to this question is one of the main contributions of this paper. Additionally, we present an efficient framework for solving a fundamental problem in 3D shape analysis: How to compute geodesic paths between given parameterized surfaces under the chosen metric?

As a motivating example, consider the three toy heart surfaces in Figure 2. These surfaces have the same shape but different parameterizations (displayed next to the surfaces). A framework not invariant to parameterization would result in a non-zero distance between these surfaces despite their shapes being identical.

**1.1 Past and Current Methods**

Similar to curves, there have been several analogous representations of surfaces. Many groups have proposed methods for studying the shapes of surfaces by embedding them in volumes and deforming these volumes under the LDDMM framework [13], [17], [8], [7], [34]. While these methods are both prominent and pioneering in medical image analysis, they are typically computationally expensive since they try to match not only the objects of interest but also some background space containing them. An alternative approach is based on manually-generated landmarks under the Kendall shape theory [10] and active shape models [6]. Others study 3D shape variabilities using level sets [26], curvature flows [16], or point cloud matching via the iterative closest point algorithm [1]. Also, there has been remarkable success in the use of medial representations for shape analysis, especially in medical image analysis, see e.g. [3], [12].

However, the most natural representation for studying shapes of 3D objects seems to be parameterized surfaces. In case of parameterized surfaces, there is an additional issue of handling the parameterization variability. Some papers, e.g. using SPHARM [20], [4] or SPHARM-PDM [33], [11], tackle this problem by choosing a fixed parameterization that is analogous to the arc-length parameterization on curves. Kilian et al. [21] presented a technique for computing geodesics between triangulated meshes (discretized surfaces) but at their given parameterizations. Similar to the elastic representations of curves, we would like to include the parameterization variable in the analysis. This inclusion results in an improved registration of features across surfaces. Of course, the question is: How can we include the parameterization variable in our shape analysis? A large set of papers in the literature treat parameterization (or registration) as a pre-processing step [35]. In other words, they take a set of surfaces and use some energy function, such as the entropy [5] or the minimum description length [9], to register points across surfaces. Once the surfaces are registered, they are compared using standard procedures. There are several fundamental problems with this approach. Firstly, the energy used for registration does not lead to a proper distance on the shape space of surfaces. Secondly, due to a registration procedure based on ensembles, the distance between any two shapes ends up being dependent on the other shapes in the ensemble. Also, the registration and the comparisons of surfaces end up being disjoint procedures and under different metrics. This certainly lacks the formalism needed to define proper distances. The contrast between these methods and our approach is presented in Figure 2.

To the best of our knowledge, there are very few techniques in the literature on a Riemannian shape analysis of parameterized surfaces that can provide geodesic paths and be invariant to re-parameterization.
1.2 Our Approach

To discuss our approach, let \( f_1 \) and \( f_2 \) denote two surfaces; \( f_1 \) and \( f_2 \) are elements of an appropriate space \( \mathcal{F} \), which is made precise later, and let \( \langle \cdot, \cdot \rangle \) be the chosen Riemannian metric on \( \mathcal{F} \). Then, under certain conditions, the geodesic distance between shapes of \( f_1 \) and \( f_2 \) will be given by quantities of type:

\[
\min_{\gamma, O} \left( F : [0,1] \to \mathcal{F}, \quad F(0) = f_1, \quad F(1) = O(f_2 \circ \gamma) \right),
\]

(1)

(This assumes that translation and scaling variability has already been removed.) Here \( F(t) \) is a parameterized path in \( \mathcal{F} \), and the quantity \( \int_0^1 \langle \langle \Gamma F(t), F(t) \rangle \rangle^{(1/2)} dt \) denotes the length of \( F \), \( L(F) \). The minimization inside the brackets, thus, denotes the problem of finding a geodesic path (locally the shortest path) between the surfaces \( f_1 \) and \( O(f_2 \circ \gamma) \), where \( O \) and \( \gamma \) stand for an arbitrary rotation and re-parameterization of \( f_2 \), respectively. The minimization outside the bracket seeks the optimal rotation and re-parameterization of the second surface so as to best match it with the first surface. In simple words, the outside optimization solves the registration or matching problem while the inside optimization solves for both an optimal deformation (geodesic) and a formal distance (geodesic distance) between shapes. An important strength of this approach is, thus, that the registration and comparison are solved jointly rather than sequentially. Another strength is that this framework can be easily extended to different types of surfaces.

The rest of this paper is organized as follows. Section 2 describes a convenient mathematical representation of embedded surfaces and introduces a parameterization-invariant Riemannian metric for shape analysis of such surfaces. It establishes the pre-shape space, the shape-preserving transformation groups, and the shape space as a quotient space of the pre-shape space. While Section 3 presents a path-straightening method for finding geodesics in the pre-shape space, Section 4 presents a registration method and an algorithm for finding geodesics in shape spaces. Section 5 presents several illustrations of these ideas using different surfaces.

2 Mathematical Representation

Let \( S \) denote a 2D smooth surface with genus zero. We will represent the surface \( S \) with its embedding \( f : \mathbb{S}^2 \to \mathbb{R}^3 \). The function \( f \) is also called a parameterization of \( S \), parameterized by elements of \( \mathbb{S}^2 \). Let the set of parameterized surfaces be \( \mathcal{F} = \{ f : \mathbb{S}^2 \to \mathbb{R}^3 \mid \int_{\mathbb{S}^2} \| f(s) \|^2 ds < \infty \text{ and } f \text{ is smooth} \} \), where \( ds \) is the standard Lebesgue measure on \( \mathbb{S}^2 \). We choose the natural Riemannian structure in the tangent space, \( T_f(\mathcal{F}) \); for any two elements \( m_1, m_2 \in T_f(\mathcal{F}) \), define an inner product: \( \langle m_1, m_2 \rangle = \int_{\mathbb{S}^2} \langle m_1(s), m_2(s) \rangle ds \), where the inner product inside the integral is the standard Euclidean product. The resulting \( L^2 \) distance between any two points \( f_1, f_2 \in \mathcal{F} \) is \( \left( \int_{\mathbb{S}^2} \| f_1(s) - f_2(s) \|^2 ds \right)^{1/2} \), and the geodesic path connecting them in \( \mathcal{F} \) is a “straight line”: \( \beta(t) = tf_2 + (1-t)f_1 \). One can represent surfaces as elements of \( \mathcal{F} \) as stated here and use the \( L^2 \) distance to compare shapes of surfaces. Although this framework is very common and seemingly convenient, it is not suitable for analyzing shapes of surfaces as it is not invariant to re-parameterizations.

We explain this point further. Let \( \Gamma \) be the set of all diffeomorphisms of \( \mathbb{S}^2 \). This set will act as the re-parameterization group for surfaces. \( \Gamma \) is a Lie group with the composition as the group operation and the identity mapping as the identity element \( \gamma_{id} \). The natural action of \( \Gamma \) on \( \mathcal{F} \) is on the right by composition: for a \( \gamma \in \Gamma \), \( f \in \mathcal{F} \), the re-parameterized surface is given by \( f \circ \gamma \). In order to unify all representations of a surface, we define the orbit of \( f \), under the action of \( \Gamma \) as: \( \{ f \circ \gamma \mid \gamma \in \Gamma \} \). The quotient space \( \mathcal{F}/\Gamma \) is the set of all such orbits, and we would like to put a natural metric on it. If \( \Gamma \) acts on \( \mathcal{F} \) by isometries, this would be feasible. The \( L^2 \) metric would simply descend to a metric on the quotient space. So we check the isometry condition:

\[
\| f_1 \circ \gamma - f_2 \circ \gamma \| = \left( \int_{\mathbb{S}^2} \| f_1(\gamma(s)) - f_2(\gamma(s)) \|^2 ds \right)^{1/2} = \left( \int_{\mathbb{S}^2} \| f_1(\hat{s}) - f_2(\hat{s}) \|^2 J_{\gamma}^{-1}(\hat{s}) ds \right)^{1/2} \neq \| f_1 - f_2 \|,
\]

where \( J_\gamma(s) \) is the Jacobian of \( \gamma \) at \( s \). This inequality comes from the fact that \( \gamma \), in general, is not area preserving and hence the Jacobian is not one at all points. This lack of isometry means that the \( L^2 \) distance between any two surfaces will not be the same if they are re-parameterized by the same element of \( \Gamma \). This also implies degeneracy, that is, it may be possible to carefully choose \( \gamma \) such that the \( L^2 \) distance between any two surfaces in \( \mathcal{F} \) is arbitrarily close to zero. One solution is to restrict to only those re-parameterizations that are area-preserving or some subset of these [15]. However, this is a severe restriction and may not be able to provide a good matching of surfaces and thus an unreliable measure of their differences. Our approach is to define
a new metric in the space $\mathcal{F}$ such that the isometry condition under the action of the re-parameterization group is satisfied.

To endow $\mathcal{F}$ with a Riemannian metric, we begin by defining a new representation of surfaces [24], [23]:

**Definition 1.** Define the mapping $Q : \mathcal{F} \rightarrow \mathbb{L}^2$ as $Q(f)(s) = \sqrt{\|a(s)\|} f(s)$, where $\|a(s)\| = \|f_x(s) \times f_y(s)\|$ is the area multiplication factor of $f$ at $s = (x, y) \in \mathbb{S}^2$.

Here $\| \cdot \|$ denotes the standard 2-norm of a vector in $\mathbb{R}^3$. The factor $\|a(s)\|$ is the ratio of infinitesimal areas of the surface at $f(s)$ and the domain at $s$. For any $f \in \mathcal{F}$, we will refer to $q(s) \equiv Q(f)(s)$ as the $q$-map of $f$. Since $\mathcal{F}$ is a set of smooth surfaces, the set of all $q$-maps is a subset of $\mathbb{L}^2(\mathbb{S}^2, \mathbb{R}^3)$, henceforth denoted by $\mathbb{L}^2$. Figure 3(a) displays the mapping $Q$ between $\mathcal{F}$ and $\mathbb{L}^2$. We can now define a new action of $\Gamma$ on $\mathbb{L}^2$, the space of $q$-maps, as follows:

![Diagram](fig3.png)

Fig. 3. (a) Mapping $Q : \mathcal{F} \rightarrow \mathbb{L}^2$, (b) Its differential at $f$, $Q_{*f} : T_f(\mathcal{F}) \rightarrow \mathbb{L}^2$.

**Definition 2.** Define a right action of $\Gamma$ on $\mathbb{L}^2$ by $\mathbb{L}^2 \times \Gamma \rightarrow \mathbb{L}^2$ as $(q, \gamma) = \sqrt{\mathcal{J}_\gamma}(q \circ \gamma)$.

An important fact about the map $Q$ is that if we reparameterize a surface by $\gamma$ and then obtain its $q$-map (Definition 1), or if we obtain its $q$-map (Definition 1) and then act by $\gamma$ (Definition 2), the result will be the same. In other words, the diagram in Figure 4 is commutative. The proof of this statement follows.

**Proof:** First, we focus on computing $a_\gamma = \frac{\partial(f \circ \gamma)}{\partial x} \times \frac{\partial(f \circ \gamma)}{\partial y}$. Let $\gamma(x, y) = (\gamma_1(x, y), \gamma_2(x, y))^T$ and $f(x, y) = (f_1(x, y), f_2(x, y), f_3(x, y))^T$. Then,

$$\frac{\partial(f \circ \gamma)}{\partial x} = \frac{\partial f}{\partial \gamma_1} \frac{\partial \gamma_1}{\partial x} + \frac{\partial f}{\partial \gamma_2} \frac{\partial \gamma_2}{\partial x}$$

and similarly, $\frac{\partial(f \circ \gamma)}{\partial y} = \frac{\partial f}{\partial \gamma_1} \frac{\partial \gamma_1}{\partial y} + \frac{\partial f}{\partial \gamma_2} \frac{\partial \gamma_2}{\partial y}$. We can write

$$\frac{\partial(f \circ \gamma)}{\partial x} = \left( f_{x, 1}^1 + f_{x, 2}^2 + f_{x, 3}^3 \right) \cdot \frac{\partial f}{\partial y} = \left( \frac{\partial f_{y, 1}}{\partial y} + \frac{\partial f_{y, 2}}{\partial y} + \frac{\partial f_{y, 3}}{\partial y} \right).$$

We can now compute $a_\gamma$, which yields,

$$a_\gamma = a \mathcal{J}_\gamma,$$

where $a$ is the Jacobian of $\gamma$.

Given this, we can verify the validity of Definition 2:

$$(q, \gamma) = \sqrt{\|a\|} (f \circ \gamma) = \sqrt{\mathcal{J}_\gamma} \sqrt{\|a\|} (f \circ \gamma) = \sqrt{\mathcal{J}_\gamma} (q \circ \gamma).$$

We can now compute $\sqrt{\|a\|}$ and obtain $q$, denoted by $Q_{*f}$. This is a linear mapping between tangent spaces $T_f(\mathcal{F})$ and $\mathbb{L}^2$ as shown in Figure 3(b). For a tangent vector $\mathbf{v} \in T_f(\mathcal{F})$ and $r \in \mathbb{R}$, the mapping $Q_{*f}$ is given by:

$$Q_{*f}(\mathbf{v}) = \frac{d}{dr}_{r=0} Q(f + rv) = \frac{1}{2 \sqrt{\|a\|}} (\|a_f + rv\|) \left|_{r=0} f + \sqrt{\|a\|} \mathbf{v} \right. \quad (2)$$

Since $\|a_f + rv\| \left|_{r=0} = \frac{\|a_f\|}{\|a\|}$, we obtain

$$Q_{*f}(\mathbf{v}) = \frac{1}{2 \|a\|^2} (a \cdot a_v) f + \sqrt{\|a\|} \mathbf{v}. \quad (3)$$

In this equation $a$ depends only on $f$ while $a_v$ depends on both $f$ and $v$. We use this differential of $Q$ to define a Riemannian metric on $\mathcal{F}$ as follows.

**Definition 3.** For any $f \in \mathcal{F}$ and any $v_1, v_2 \in T_f(\mathcal{F})$, define the inner product:

$$(v_1, v_2)_f = (Q_{*f}(v_1), Q_{*f}(v_2)). \quad (4)$$

where the inner product on the right side is the standard inner product in $\mathbb{L}^2$.

With this induced metric, $\mathcal{F}$ becomes a Riemannian manifold and we want to compute geodesic distances between two points, say $f_1$ and $f_2$, in $\mathcal{F}$. To write the metric in Definition 3 in full detail, we use the expression
for $Q_{*,f}(v)$ given in Eqn. 3:
\[
\langle\langle v_1, v_2\rangle\rangle_f = \langle Q_{*,f}(v_1), Q_{*,f}(v_2)\rangle_f \\
= \langle \frac{1}{2\|a\|^2} (a \cdot a v_1) f + \sqrt{\|a\|} \, v_1, \frac{1}{2\|a\|^2} (a \cdot a v_2) f + \sqrt{\|a\|} \, v_2 \rangle_f \\
= \langle \frac{1}{4\|a\|^2} (a \cdot a v_1) f, (a \cdot a v_2) f \rangle_f + \langle \frac{1}{2\|a\|^2} [(a \cdot a v_1) v_2 + (a \cdot a v_2) v_1], f \rangle_f + \langle \|a\| v_1, v_2 \rangle_f.
\]

An important property of this metric is that the action of $\Gamma$ on $\mathcal{F}$ is by isometries.

**Proposition 1.** Given a surface $f \in (\mathcal{F})$, a $\gamma \in \Gamma$ and two tangent vectors $v_1, v_2 \in T_f(\mathcal{F})$:
\[
\langle\langle v_1 \circ \gamma, v_2 \circ \gamma\rangle\rangle_{f \circ \gamma} = \langle\langle v_1, v_2\rangle\rangle_f. \tag{5}
\]

**Proof:** Given a surface $f \in \mathcal{F}$, a $\gamma \in \Gamma$ and two tangent vectors $v_1, v_2 \in T_f\mathcal{F}$, we need to show that,
\[
\langle\langle v_1 \circ \gamma, v_2 \circ \gamma\rangle\rangle_{f \circ \gamma} = \langle\langle v_1, v_2\rangle\rangle_f.
\]
We begin by showing that, $Q_{*,(f \circ \gamma)}(v \circ \gamma) = Q_{*,f}(v)$ and the rest of the proof will follow:
\[
Q_{*,(f \circ \gamma)}(v \circ \gamma) = \frac{1}{2\|a\|^2} (a \cdot a_{\gamma, v \circ \gamma}) f \circ \gamma + \sqrt{\|a\|} \, (v \circ \gamma) \\
= \frac{1}{2\|a\|^2} (a \cdot a_{\gamma}) J_{\gamma} f \circ \gamma + \sqrt{\|a\|} \, J_{\gamma} (v \circ \gamma) \\
= \frac{1}{2\|a\|^2} (a \cdot a_{\gamma}) \sqrt{J_{\gamma}} f \circ \gamma + \sqrt{\|a\|} \, \sqrt{J_{\gamma}} (v \circ \gamma) \\
= \frac{1}{2\|a\|^2} (a \cdot a_{\gamma}) f + \sqrt{\|a\|} \, v = Q_{*,f}(v).
\]

Thus, $\langle\langle v_1 \circ \gamma, v_2 \circ \gamma\rangle\rangle_{f \circ \gamma}$ equals
\[
\langle Q_{*,(f \circ \gamma)}(v_1 \circ \gamma), Q_{*,(f \circ \gamma)}(v_2 \circ \gamma)\rangle_{L^2} = \langle Q_{*,f}(v_1), Q_{*,f}(v_2)\rangle_{L^2} = \langle\langle v_1, v_2\rangle\rangle_f.
\]

\[\Box\]

2.2 Pre-Shape and Shape Space

Shape analysis of surfaces can be made invariant to certain global transformations by normalizing. The translation of surfaces is easily taken care of by centering:
\[
f_{\text{centered}}(s) = f(s) - \frac{\int_{s_2} f(s) \|a(s)\| ds}{\int_{s_2} \|a(s)\| ds}.
\]
Scaling can be removed by re-scaling all surfaces to have unit area,
\[
f_{\text{scaled}}(s) = \frac{f(s)}{\|f(s)\|}.
\]

With a slight abuse of notation, we define the space of normalized surfaces as $\mathcal{F}$. $\mathcal{F}$ forms the pre-shape space in our analysis. The remaining groups – rotation and re-parameterization – are dealt with differently, by removing them algebraically from the representation space.

1) **Rotation Group**, $SO(3)$: The rotation group $SO(3)$ acts on $\mathcal{F}$, $SO(3) \times \mathcal{F} \to \mathcal{F}$ according to $(O,f) = Of$, for $O \in SO(3)$ and $f \in \mathcal{F}$. It is easy to check that the action of $SO(3)$ on $\mathcal{F}$ under the induced metric is by isometries.

2) **Re-Parameterization Group**, $\Gamma$: The re-parameterization group $\Gamma$ acts on $\mathcal{F}$, $\Gamma \times \mathcal{F} \to \mathcal{F}$ according to $(f,\gamma) = (f \circ \gamma)$. As discussed in Section 2.1 the re-parametrization group $\Gamma$ acts on $\mathcal{F}$ by isometries under the induced metric.

**Proposition 2.** The actions of $\Gamma$ and $SO(3)$ on $\mathcal{F}$ commute.

Since the actions of $SO(3)$ and $\Gamma$ commute we can define an action of the product of the groups on $\mathcal{F}$. The orbit of a surface $f$ is given by:
\[
[f] = \text{closure}(O(f \circ \gamma) | O \in SO(3), \gamma \in \Gamma) \tag{6}
\]
and the set of all $[f]$ is defined to be $S = \{[f]|f \in \mathcal{F}\}$. Since the orbits under $\Gamma$ are not closed, we use their closures to define equivalence classes.

The next step is to define geodesic paths in $\mathcal{F}$ and $S$. We start with the case of $\mathcal{F}$; the geodesic distance between any two points $f_1$, $f_2 \in \mathcal{F}$, $d_\mathcal{F}(f_1, f_2)$, is given by:
\[
\min_{F: [0,1] \to \mathcal{F}} \int_0^1 \langle\langle F(t), F(t)\rangle\rangle^{(1/2)} dt. \tag{7}
\]

We will use a path-straightening approach for solving this problem in Section 3. Once we have an algorithm for finding geodesics in $\mathcal{F}$, we can obtain geodesics and geodesic lengths in $S$ by solving an additional minimization problem over $SO(3) \times \Gamma$ as stated in Eqn. 1. This problem searches over the orbit $[f_2]$ so that the geodesic distance between $f_1$ and an element of $[f_2]$ is minimized. We will use a gradient-based approach to solve this problem in Section 4.

3 GEODESICS IN THE PRE-SHAPE SPACE $\mathcal{F}$

Consider the problem of finding geodesics between surfaces $f_1$ and $f_2$ in $\mathcal{F}$ using a path-straightening approach. This method was first described in [22] and was later used for finding geodesics between elastic curves in [32]. The basic idea here is to connect $f_1$ and $f_2$ by any initial path, e.g. using a straight line under the $L^2$ metric, and then iteratively “straighten” it until it becomes a geodesic. This update is performed using the gradient of an appropriate energy function. Earlier works on path-straightening involved non-linear manifolds inside a larger vector space such that the Riemannian metric was a restriction of the standard metric on the larger space. As seen next, the current case is different. The space of parameterized surfaces is a vector space, but the metric on this space is non-standard.
Let $F : [0, 1] \to \mathcal{F}$ denote a path in $\mathcal{F}$. The energy of the path $F$ under the induced metric is defined to be:

$$E[F] = \int_0^1 \langle (F_t), F_t \rangle_{\mathcal{F}} dt$$

Using Defn. 3

$$= \int_0^1 \langle Q_{*, F}(F_t), Q_{*, F}(F_t) \rangle dt$$

$$= \int_0^1 \left( \frac{1}{\|A\|^2} (A \cdot A_t)^2 (F, F) \right)$$

$$+ \langle (A \cdot A_t) F_t, F_t \rangle + \langle \|A\| F_t, F_t \rangle dt$$

$$= \int_0^1 \int_{\mathbb{R}^3} \frac{1}{\|A\|^2} (A \cdot A_t)^2 (F, F)$$

$$+ \frac{1}{\|A\|} (A \cdot A_t) (F_t, F_t)$$

$$+ \frac{1}{\|A\|} (F_t, F_t)$$

$$+ \frac{1}{\|A\|} (A \cdot A_t) dt$$

In this derivation we have suppressed the argument $t$ for all the quantities. Also, we use $A(t)$ to imply $a(F_t(t))$. It is well known that a critical point of $E$ is a geodesic path in $\mathcal{F}$. To find a critical point, we are going to use the gradient $\nabla E_F$ which, in turn, is approximated using directional derivatives, $\nabla E_F(G)$, where $G \in \mathcal{G}$ is a perturbation of the path $F$. Here $\mathcal{G}$ denotes the set of all possible perturbations of $F$. Figure 5 is a depiction of this approach. We start with an initial path $F$ and iteratively update it in the direction of $\nabla E$ until we arrive at the critical point $F^*$, which is the desired geodesic.

![Fig. 5. An example of a geodesic path in $\mathcal{F}$ via path-straightening: initial path (left) and final path (middle) and energy evolution (right).](image)

### 3.1 Directional Derivative of $E$

In this section, we provide some details about the derivation of $\nabla E_F(G)$, This directional derivative will be used to approximate the gradient of $E$. The derivative of $E$ in the direction $G$ is given by $\nabla E_F(G) = \frac{d}{dt} E(F + \epsilon G)|_{\epsilon = 0}$. The energy of the perturbed path is:

$$E[F + \epsilon G] = \int_0^1 \langle Q_{*, F + \epsilon G}(F_t + \epsilon G_t), Q_{*, F + \epsilon G}(F_t + \epsilon G_t) \rangle dt.$$

In order to write down an analytical formula for the energy gradient, we first write down and derive some terms that are needed later. All of these terms involve $F$ and are, therefore, functions of $t$, $x$, and $y$. These arguments are suppressed for the sake of brevity. Since $A = F_x \times F_y$ and $A_t = F_{t,x} \times F_y + F_x \times F_{t,y}$, we have:

- $\delta_G A \equiv G_x \times F_y + F_x \times G_y$
- $\delta_G A_t \equiv G_{t,x} \times F_y + F_{t,x} \times G_y + G_x \times F_{t,y} + F_x \times G_{t,y}$

Here we use the notation $\delta_G A$ to represent the directional derivative of the function $A$ in the direction $G$. Similarly, we can obtain the directional derivatives of other terms:

- $\delta_G (F \cdot F) = 2(F \cdot G)$
- $\delta_G (F_t \cdot F_t) = 2(F_t \cdot G_t)$
- $\delta_G (F \cdot F_t) = (F_t \cdot G)(F \cdot G_t)$
- $\delta_G \|A\| = (A \cdot \delta_G A)$
- $\delta_G (A \cdot A_t) = (A \cdot \delta_G A_t)(A_t \cdot \delta_G A)$

We are now ready to write down the analytic formula for the path-straightening energy gradient, which is given by:

$$\nabla E_F(G) = \frac{d}{d\epsilon} E(F + \epsilon G)|_{\epsilon = 0} = \int_0^1 \int_{\mathbb{R}^3} \left( -3 \frac{(A \cdot \delta_G A)^2 (F, F)}{4\|A\|^5} ight)$$

$$+ \frac{1}{2\|A\|^3} (A \cdot A_t) (A \cdot \delta_G A_t + A_t \cdot \delta_G A) (F_t, F_t)$$

$$+ \frac{1}{2\|A\|^3} (A \cdot \delta_G A) (A_t, G + F \cdot G_t)$$

$$+ \frac{1}{\|A\|} (A \cdot \delta_G A) (F_t, F_t)$$

$$+ \frac{1}{\|A\|} (F_t, G + F \cdot G_t)$$

$$+ \frac{1}{\|A\|} (A \cdot \delta_G A) (F_t, F_t)$$

$$+ \frac{1}{\|A\|} (A \cdot \delta_G A_t) (F_t, F_t)$$

The terms inside the integrals may be re-written as $H_1 \cdot \delta_G A + H_2 \cdot \delta_G A_t + H_3 \cdot G + H_4 \cdot G_t$, where,

$$H_1 = \frac{A}{\|A\|} \left( \frac{-3}{4\|A\|^5} (A \cdot A_t)^2 (F, F) - \frac{1}{\|A\|} (A \cdot A_t) (F_t, F_t) + (F_t, F_t) \right)$$

$$+ \frac{A}{\|A\|} (A \cdot A_t) (F_t, F_t) + (F_t, F_t)$$

$$H_2 = \frac{A}{2\|A\|^3} (A \cdot A_t)(F_t, F)$$

$$H_3 = \frac{A}{\|A\|} (A \cdot A_t)^2 F + \frac{A}{\|A\|} (A \cdot A_t) F_t$$

$$H_4 = \frac{A}{\|A\|} (A \cdot A_t) F + 2\|A\| F_t$$

Furthermore, we can re-write terms $H_1 \cdot \delta_G A$ and $H_2 \cdot \delta_G A_t$ and combine them as follows:

$$H_1 \cdot \delta_G A = H_1 \cdot (F_x \times G_y) + (G_x \times F_y) \cdot H_1$$

$$+ (H_1 \cdot F_x) \cdot G_y + (F_y \times H_1) \cdot G_x$$

$$H_2 \cdot \delta_G A_t = (G_{t,x} \times F_y) \cdot H_2 + H_2 \cdot (F_{t,x} \times G_y)$$

$$+ (G_x \times F_{t,y}) \cdot H_2 + H_2 \cdot (F_x \times G_{t,y})$$

$$+ (F_y \times H_2) \cdot G_{t,x} + (H_2 \times F_{t,x}) \cdot G_y$$

$$+ (F_{t,y} \times H_2) \cdot G_x + (H_2 \times F_x) \cdot G_{t,y}$$
Thus, the final expression for the directional derivative of $E$ is:

$$
\nabla E_F(G) = \int_0^1 \int_{S^2} (H_3 \cdot G + H_4 \cdot G_t + M_3 \cdot G_x) ds dt
+ M_4 \cdot G_y + M_5 \cdot G_{t,x} + M_6 \cdot G_{t,y}) ds dt.
$$

### 3.2 Orthonormal Basis of $G$

In order to approximate $\nabla E_F$, we utilize the directional derivative of $E$. To compute this derivative, we use an orthonormal basis of $G$, $P = \{p_i|i = 1, 2, \ldots\}$, and set $\nabla E_F = \sum_{i=1}^N (\nabla E_F(p_i)) p_i$. The next question is: How can we form a basis for $G$? Each perturbation $G : S^2 \times [0, 1] \to \mathbb{R}^3$ has three arguments, $x, y, t$, where $x$ and $y$ are the coordinates on $S^2$, and $t$ is the time index along the path. We begin by defining two bases $\hat{P}^s : S^2 \to \mathbb{R}$, $\hat{P}^t : [0, 1] \to \mathbb{R}$. There is an additional restriction on $\hat{P}^t$, that is, $\hat{P}^t(0) = 0$, $\hat{P}^t(1) = 0$ because we do not want to perturb the starting and the end points of the path $F$. In order to define $\hat{P}^s$ we utilize the spherical harmonics functions. It is well known that any square integrable function on $S^2$ can be expressed as a linear combination of spherical harmonics. The basis $\hat{P}^t$ is defined as follows: $\hat{P}^t = \{ \sin(2\pi it) \cdot \cos(2\pi it) - 1 | 1 \leq i \leq c_3, i \in \mathbb{Z} \}$. We use all possible products of $\hat{P}^s$, $\hat{P}^t$ to form a basis $\hat{P} : S^2 \times [0, 1] \to \mathbb{R}$. The final step is to define the full basis $P : S^2 \times [0, 1] \to \mathbb{R}^3$ by utilizing three copies of $\hat{P}$:

$$
P(x, y, t) = \begin{bmatrix}
\hat{P}(x, y, t) \\
\hat{P}(x, y, t) \\
\hat{P}(x, y, t)
\end{bmatrix}.
$$

We orthonormalize this basis using the Gram-Schmidt procedure under the following metric (given two basis elements $G^1, G^2 \in G$):

$$
(G^1, G^2) = \int_0^1 \int_{S^2} (G^1 \cdot G^2 + G^1_{\phi} \cdot G^2_{\phi} + G^1_{\theta} \cdot G^2_{\theta}) ds dt
+ G^1_y \cdot G^2_y + G^1_{t,x} \cdot G^2_{t,x} + G^1_{t,y} \cdot G^2_{t,y}) ds dt.
$$

In practice, we use a subset containing 1400 basis elements of $P$ to approximate $\nabla E_F(G)$.

The accuracy of this path-straightening algorithm depends on the number of basis elements used to approximate the gradient. As we increase the number of basis elements, the approximation of the geodesic path improves. In Figure 6, we plot the distance in $F$ between the two displayed surfaces as a function of the degree of spherical harmonics used in the basis construction. In this example, we fix the value of $c_3$ to 2. We note that as the degree increases, the distance between the surfaces decreases and eventually stabilizes. The same holds when we increase the value of $c_3$, while holding the degree of spherical harmonics constant. This factor seems to play a smaller role in the accuracy of the geodesic computation.

![Fig. 6.](image.png)

(a) Decrease in the distance as the degree of spherical harmonics increases from 1 to 12 (the factor $c_3$ is kept constant at 2). (b) Decrease in the distance as the factor $c_3$ increases from 1 to 5 (the degree of spherical harmonics is kept constant at 3).

Now we show some examples of geodesics in $F$ obtained using path-straightening. To demonstrate the effectiveness of path-straightening, we consider a special case where $f_1 = f_2 = f$ and initialize a path where $F(t) \neq f$ for $t$ in the interior of the path. Of course, we expect the geodesic path to be $F^*(t) = f$, a constant path. The results are displayed in Figure 7. Using path-straightening, we obtain a 91.4% decrease in the energy function, and the resulting path is visibly the same surface. When we increased the number of basis elements used to approximate $\nabla E$, we found the energy decrease to be even greater. These paths are computed and displayed by discretizing at times $t_i = (i - 1)/6, i = 1, 2, \ldots, 7$.

Some additional examples of geodesics in $F$ are presented later in Section 4.4. Once we have a geodesic path $F^*$ between any two points, the distance in the pre-shape space between $f_1$ and $f_2$, $d_F(f_1, f_2)$, is the length of $F^*$, as specified in Eqn. 7.

### 4 Geodesics in Shape Space $S$

Now, we consider the problem of finding geodesics between surfaces in $S$. This requires solving an additional optimization over the product $SO(3) \times \Gamma$. 

$$
H_1 \cdot \delta C A + H_2 \cdot \delta C A_t = (F_y \times H_2) \cdot G_{t,x}
+ (H_2 \times F_{t,x} + H_1 \times F_x) \cdot G_y
+ (F_{t,y} \times H_2 + F_y \times H_1) \cdot G_x
+ (H_2 \times F_x) \cdot G_{t,y}.
$$

We now have:

$$
(F_y \times H_2) \cdot G_{t,x} + (H_2 \times F_{t,x} + H_1 \times F_x) \cdot G_y
+ (F_{t,y} \times H_2 + F_y \times H_1) \cdot G_x + (H_2 \times F_x) \cdot G_{t,y}
= H_3 \cdot G + H_4 \cdot G_t + M_3 \cdot G_x + M_4 \cdot G_y
+ M_5 \cdot G_{t,x} + M_6 \cdot G_{t,y}.
$$

Thus, the final expression for the directional derivative of the energy function is:
4.1 Optimization Over Rotation

We can use a gradient approach for this optimization, but instead we will use an approximate albeit efficient technique based on Procrustes analysis. For a fixed $\gamma \in \Gamma$, the minimization over $SO(3)$ is performed as follows. Compute the $3 \times 3$ matrix $C = \int_{[f_2]} f_1(s) f_2(s)^T ds$. Then, using the singular value decomposition $C = U \Sigma V^T$, we can define the optimal rotation as $O^* = UV^T$ (if the determinant of $C$ is negative, the last column of $V^T$ changes sign).

4.2 Optimization Over Re-Parameterization

In order to solve the optimization problem over $\Gamma$ in Eqn. 1, we will use a gradient approach. Although this approach has an obvious limitation of converging to a local solution, it is still general enough to be applicable to general cost functions. Additionally, we have tried to circumvent the issue of a local solution by taking multiple initializations. This is similar to the gradient approach taken in Kurtek et al. [24], [23]; the difference lies in the cost function used for optimization. In earlier papers, we address a problem of the type $\min_{\gamma \in \Gamma} \|q_1 - (q_2, \gamma)\|^2$, where $q_1$ and $q_2$ are $q$-maps of $f_1$ and $f_2$ and here we minimize a cost function of the type $d_F(f_1, f_2 \circ \gamma)^2$.

We begin by defining a map $\psi: \Gamma \rightarrow [f_2]$ by $\psi(\gamma) = \tilde{f}_2 \circ \gamma$, for a fixed $\tilde{f}_2 \in [f_2]$. Figure 8(a) shows this map pictorially. The cost function for optimization relates to the quantity inside the parenthesis in Eqn. 1. Rather than taking that quantity itself, we minimize its square for better numerical stability. Using Eqn. 7, we can define the cost function to be: $H : \Gamma \rightarrow \mathbb{R}$,

$$H[\gamma] = d_F(f_1, \tilde{f}_2 \circ \gamma)^2 = d_F(f_1, \psi(\gamma))^2,$$

where $\tilde{f}_2 = f_2 \circ \gamma_0$, and $\gamma_0$, and $\gamma$ denote the current and the incremental re-parameterizations, respectively. If we have an orthonormal basis for $T_{\gamma_{id}}(\Gamma)$ and the differential of $\psi$, we can compute the full gradient of $H$ at $\gamma_{id}$ and use it to update $\gamma_0$. This leaves two remaining issues: (1) the specification of an orthonormal basis of $T_{\gamma_{id}}(\Gamma)$ and (2) the derivation of the differential $\psi_{*,\gamma_{id}}$.

The tangent space of $\Gamma$ at identity $\gamma_{id}$ is:

$$T_{\gamma_{id}}(\Gamma) = \{ b : S^2 \rightarrow T(S^2) | b \text{ is a smooth vector field and is tangential to } S^2 \}.$$  

We are going to construct an orthonormal basis for $T_{\gamma_{id}}(\Gamma)$. For this purpose, we use gradients of spherical harmonics. We denote the full orthonormal basis sets as $\mathcal{B}$. We show four examples of this basis in Figure 9. (Further details can be found in Kurtek et al. [23].)

Now we determine the differential of the group action $\psi$ at the identity, $\psi_{*,\gamma_{id}} : T_{\gamma_{id}}(\Gamma) \rightarrow T_{f_2}([f_2])$.

**Proposition 3.** Let $b$ be a tangent vector field on $S^2$, i.e., an element of $T_{\gamma_{id}}(\Gamma)$, let $f_2 = (f_2^1, f_2^2, f_2^3)$, where each $f_2^j$ is a real-valued function on $S^2$. Then, $\psi_{*,\gamma_{id}}(b)$ will also have three components, $(\psi_{*,\gamma_{id}}^1, \psi_{*,\gamma_{id}}^2, \psi_{*,\gamma_{id}}^3)$, given by the formula:

$$\psi_{*,\gamma_{id}}^j(b) = \nabla \tilde{f}_2^j \cdot b, \quad j = 1, 2, 3,$$

where $\nabla \tilde{f}_2^j$ is the gradient of $\tilde{f}_2^j$.
Figure 8(b) is a pictorial depiction of the map \( \psi_{s_1 \to s_2} \) between tangent spaces of \( \Gamma \) and \( F \).

Now the full gradient of \( H \) with respect to \( \gamma \) is an element of \( T_{\gamma_0} \Gamma \) given by:
\[
d\gamma = \sum_{b_i \in B} \langle (F_1(1), \psi_{s_1 \to s_2}(b_i)) F(1) b_i \rangle
\]
where \( F \) is the current geodesic in \( F \) between \( f_1 \) and \( f_2 \). This linear combination of the orthonormal basis elements of \( T_{\gamma_0} \Gamma \) provides an incremental update of \( f_2 \) in the orbit \([f_2]\). The optimal re-parameterization \( \hat{\gamma} \) is defined as the concatenation of all incremental updates \( d\gamma \). Figure 10 gives a pictorial depiction of this algorithm. It shows the fixed surface \( f_1 \) and the search over the orbit of \( f_2 \) to reach \( f_2 \circ \hat{\gamma} \).

\[ \text{Fig. 10. Matching of surfaces through re-parameterization.} \]

### 4.3 Geodesics in \( S \)

The joint optimization results from alternating between optimizations over \( \Gamma \) and \( SO(3) \) until convergence. An important question in this gradient approach is the initialization over \( \Gamma \). We use the following strategy. First, we note that \( SO(3) \) forms a subset of the re-parameterization group, \( \Gamma \). We initialize the gradient search by utilizing the largest irreducible finite subgroup of \( SO(3) \), call it \( K \). Denote by \( k_1, k_2, \ldots, k_{60} \) the elements of \( K \); each of them acts on \( F \) according to \( f \circ k_i \). The initialization for optimization over \( \Gamma \) is a search over \( K \) by solving for:
\[
\hat{i} = \arg \min_{i=1,2,\ldots,60} d_F(f_1, \hat{O}_i(f_2 \circ k_i))^2.
\]

Here, \( \hat{O}_i \) is the optimal rotation of \( \tilde{f}_2 \circ h_i \) to best match \( f_1 \), obtained through Procrustes analysis.

Now we present an example of optimization over \( SO(3) \times \Gamma \). Figure 11 displays the result of this minimization in matching two closed surfaces with dual bumps with different placements. We display the matching between surfaces by transferring the colormap from \( f_1 \) onto the corresponding points on \( f_2 \). Thus, a good matching implies that similar features are shaded by similar colors. In the initial matching between the surfaces the bumps on the two surfaces do not match each other. But, after optimization, the grid on \( O^*(f_2 \circ \gamma^*) \) is stretched and compressed such that the two bumps on the two surfaces are matched. The stretching and compression of the grid is clearly seen in the display of \( \gamma^* \).

**Implementation:** To compute geodesics in \( S \), we need to solve two optimization problems in Eqn. 1. Although we can use the methodology described in this paper to solve for the optimal re-parameterization, \( \gamma^* \), we use an approximate but more efficient solution. It is important to note that the two procedures provide comparable results. We perform the joint optimization in two steps. Given two surfaces \( f_1 \) and \( f_2 \) as defined in previous sections, we first obtain the optimal parameterization \( \gamma^* \) of \( f_2 \) using the gradient technique described in [24]. Given the optimal parameterization of the second surface, we proceed to compute the geodesic path \( F^* \) in \( S \). The result of this procedure is an approximation of the geodesic distance and path between \( f_1 \) and \( O^*(f_2 \circ \gamma^*) \).

**Computational Cost:** For these experiments, we used the Matlab environment on an Intel Xeon CPU (2.50GHz, 8GB RAM, Windows XP). When we sample the surfaces with 2500 points and use 1400 basis elements for the path-straightening optimization, the average computational cost for computing a geodesic in \( F \) (10 iterations) is approximately 90s, and in \( S \) it is approximately 100s due to the extra computation of \( \gamma^* \).

### 4.4 Comparisons of Geodesics in \( F \) and \( S \)

In this section, we highlight the improvements in matching of surfaces during the optimization over \( SO(3) \times \Gamma \) by comparing geodesic paths between the same pairs of surfaces in \( F \) and \( S \). In all of these experiments, we notice that \( E[F^*] \) for geodesics in \( S \) is an order of magnitude smaller than the energies for geodesics in \( F \).

In Figure 12, we display a simple example of geodesic computations for two surfaces with dual bumps. We display the deformation vector field associated with the shooting vector, \( F^*(0) \), for the geodesic, \( F^*(t) \), computed in \( F \). In the case \( S \), we display the tangent vector...
field resulting from the optimization over $\Gamma$ and the deformation vector field from the subsequent geodesic computation. We can clearly see from the geodesic path and corresponding vector fields, that in the pre-shape space the two bumps on the first surface are contracted while new bumps are created. This results in the midpoint of the geodesic path having four smaller bumps. In the shape space, the effect is as if the bumps on the first surface move to the location of the bumps on the second surface. This improved geodesic is the result of improved matching due to the optimization over the re-parameterization group.

Figure 13 (top row) displays an example of geodesics between surfaces with different number and placement of peaks. The geodesic in $F$ distorts the large peak on $f_1$ as it is transformed into one of the peaks on $f_2$. On the other hand, the geodesic in $S$ preserves the surface features much better. The second row displays the geodesic paths between identical heart surfaces but with two different parameterizations, that is, $f_2 = f_1 \circ \gamma$. Since our metric is parameterization invariant, we expect the distance between them to be zero. As shown in the figure, the resulting geodesic in $S$ is a constant and the associated energy is almost zero. The third row displays the geodesics between a bottle and a chess piece. We note that the geodesic path in the shape space has much lower energy than that in the pre-shape space. Although the differences in geodesic paths in this example are more subtle, the difference in geodesic energies is significant.

The last row in Figure 14 displays the geodesic paths between surfaces of two chess pieces. The preservation of features along the geodesic in the shape space is clearly seen in this example. Figure 14 shows more examples of geodesics between closed surfaces with various shapes. The last row in Figure 14 displays the geodesic paths between surfaces of a left pallidus and a right thalamus extracted from MRI scans of a human brain. Computing shape differences using geodesic paths between surfaces of anatomical structures is a very important application of shape analysis. More natural geodesics provide us with a more accurate measure of differences between anatomical surfaces, which can be critical in disease diagnostics. Once again, the geodesic path in $S$ has much lower energy than that in $F$ and represents a more natural transformation from $f_1$ to $f_2$.

5 EXPERIMENTAL RESULTS

As mentioned earlier, the geodesic paths provide us with tools for comparing, matching, and deforming parameterized surfaces. We suggest a comparison of shapes of 3D objects using geodesic distances between their boundary surfaces in the shape space. This section presents a specific application to illustrate that idea.

Classification of Anatomical Shapes

Shape analysis of anatomical structures can play an important role in detection, classification, and monitoring of different diseases, especially those that affect the human brain. In this section, we study shapes of several subcortical brain structures (hippocampus, putamen, thalamus, caudate, etc.) of subjects and analyze their relationship to certain mathematical deficiencies exhibited by those subjects. We begin by computing the pairwise geodesic distances between corresponding substructures for 106 subjects and we consider 20 such structures for each subject. Some examples of these substructures are shown in Figure 15. We utilize a leave-one-out nearest neighbor classifier to assign a subject to the case or control group. For the most promising structures (highlighted in the table), we computed classification rates using the SPHARM-PDM framework as described in [33]. As a distance measure for this technique we computed the pairwise root mean squared distances.
The best single classifier is the left hippocampus, which achieves a 62.3% classification rate. This result is supported by previous studies, which have shown that the hippocampus is correlated with cognitive abilities and memory [36]. The other single classifiers, which yield best performance are the right thalamus (61.3%), and the left and right pallidus (58.5% and 57.5%). The left and right pallidus are major components of the basal ganglia, which along with the thalamus form a very important connection in the human brain. Previous studies have shown that the basal ganglia and its connection with the thalamus play an important role in learning and cognitive functioning [30], [27]. The pallidus has also been identified as a very important determinant in working memory, which is very closely linked to mathematics performance [2]. We compare our classification results with those obtained using the same classifier but with distances resulting from SPHARM-PDM; our method outperforms SPARHM-PDM in all of those cases.

We can improve classification rates by combining distances for individual structures. The objective here is to maximize the classification rate of disease and controls using some combination of distances based on individual structures. We exhaustively search over a discrete set of weights by taking a few structures at a time. Let us define $d_{tot} = \sum_{i=1}^{n} w_i d_i$, where $w_i \geq 0$ such that $\sum_{i=1}^{n} w_i = 1$ and $d_i$s are the geodesic distances for individual structures. We start with the single structure that provides the highest classification rate and proceed by adding new structures one at a time, using the weights $w_i = 0, 0.05, 0.1, ..., 0.95, 1$, for the new structure, such that the classification rate improves maximally at each step. We stop when no more improvement is noticed. One combination of four structures resulted in a 71.7% classification rate. This combination consisted of: 0.5682 L Hippocampus + 0.1591 R Inferior Parietal Lobe + 0.1591 R anterior cingulate gyrus + 0.1136 L Caudate. The inclusion of the inferior parietal lobe in this multiple structure classifier is supported by some evidence pre-

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<table>
<thead>
<tr>
<th>Pre-Shape Space</th>
<th>Shape Space</th>
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</thead>
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<tr>
<td><img src="image1.png" alt="Pre-Shape Space" /></td>
<td><img src="image2.png" alt="Shape Space" /></td>
</tr>
<tr>
<td>$E(F^*) = 0.0379$</td>
<td>$E(F^*) = 0.0118$</td>
</tr>
<tr>
<td><img src="image3.png" alt="Pre-Shape Space" /></td>
<td><img src="image4.png" alt="Shape Space" /></td>
</tr>
<tr>
<td>$E(F^*) = 0.0896$</td>
<td>$E(F^*) = 0.0028$</td>
</tr>
<tr>
<td><img src="image5.png" alt="Pre-Shape Space" /></td>
<td><img src="image6.png" alt="Shape Space" /></td>
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<tr>
<td>$E(F^*) = 0.0306$</td>
<td>$E(F^*) = 0.0105$</td>
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<tr>
<td><img src="image7.png" alt="Pre-Shape Space" /></td>
<td><img src="image8.png" alt="Shape Space" /></td>
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<tr>
<td>$E(F^*) = 0.0757$</td>
<td>$E(F^*) = 0.0618$</td>
</tr>
</tbody>
</table>

**Fig. 13.** Comparison of geodesics in $\mathcal{F}$ and $\mathcal{S}$, and their energies. Top: two simple surfaces with different number and placements of peaks. Second row: two heart surfaces with different parameterizations. Third row: a bottle and a chess piece. Last row: two chess pieces.
Table 1
Classification performance for individual substructures.

<table>
<thead>
<tr>
<th>Structure</th>
<th>Side</th>
<th>Caudate</th>
<th>Hippocampus</th>
<th>Pallidus</th>
<th>Inferior Parietal Lobe</th>
<th>Posterior Cingulate Gyrus</th>
<th>Insula</th>
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<tr>
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<td>L</td>
<td>R</td>
<td>L</td>
<td>R</td>
<td>L</td>
<td>R</td>
<td>L</td>
</tr>
<tr>
<td>Our Method (%)</td>
<td>51.9</td>
<td>51.9</td>
<td>62.3</td>
<td>52.8</td>
<td>58.5</td>
<td>57.5</td>
<td>57.5</td>
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<tr>
<td>SPHARM-PDM (%)</td>
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<td>-</td>
<td>42.2</td>
<td>-</td>
<td>36.6</td>
<td>30.9</td>
<td>-</td>
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<td></td>
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<td>L</td>
<td>R</td>
<td>L</td>
<td>R</td>
<td>L</td>
</tr>
<tr>
<td>Our Method (%)</td>
<td>51.9</td>
<td>61.3</td>
<td>57.5</td>
<td>53.8</td>
<td>46.2</td>
<td>56.6</td>
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</tr>
<tr>
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<td>50.0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
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</tbody>
</table>


6 Conclusion and Discussion
Shape analysis of 3D objects is very important in many scientific fields. We have proposed a novel Riemannian framework for computing geodesic paths between shapes of parameterized surfaces. These geodesics are invariant to rigid motion, scaling and most importantly re-parametrization of individual surfaces. The geodesic computation is based on a path-straightening technique that iteratively corrects paths between surfaces until geodesics are achieved. The iterative update is based on the gradient of a path energy; this gradient is approximated using a large number of basis elements in the perturbation space. We have presented some examples of geodesics between surfaces in the pre-shape and shape spaces and utilized the distances between surfaces for classification of anatomical shapes. An important application of this framework is in computations of means, covariances, and probability models to capture shape variability within shape classes [25]. There are two main limitations of this work. First, the metric used here is not invariant to translations. Second, there
is no physical interpretation of the metric induced on the space of parameterized surfaces. One would like a metric, which can be interpreted as a combination of bending and stretching. In the future, we would like to explore alternative choices for the metric, which can avoid these problems.

As stated earlier, this framework is easily extended to other types of parameterized surfaces, such as quadrilateral surfaces. Due to limited space, we have not provided details for this case. Instead, we provide some examples for illustration purposes. First, we show the effect of details for this case. Instead, we provide some examples of general surfaces. Due to limited space, we have not provided to explore alternative choices for the metric, which can be interpreted as a combination of each other. But, after applying the optimal rotation $O^*$ and re-parameterization $\gamma^*$ to $\tilde{f}_2$ the high peak on $f_1$ matches the high peak on $O^*(\tilde{f}_2 \circ \gamma^*)$ very well. In addition, we provide two examples (in Figure 17) of geodesic path computations in the shape and pre-shape space. We can draw the same conclusions from these examples as was done in Section 4.4. That is, the geodesic path energy is much smaller in the shape space due to improved feature matching across surfaces.

Fig. 16. Comparison of initial matching between $f_1$ and $\tilde{f}_2$ and matching after optimization over $\Gamma$.

REFERENCES


Fig. 17. Comparison of geodesics in $\mathcal{F}$ and $\mathcal{S}$ and their energies. Top: surfaces formed by images with two peaks each at different locations. Second row: surfaces of revolution.


**Sebastian Kurtek** Sebastian Kurtek received his BS degree in Mathematics from Tulane University in 2007, and MS degree in Statistics from the Florida State University in 2009, where he is currently a PhD candidate.

**Eric Klassen** Eric Klassen is a professor of mathematics at Florida State University. He earned his PhD in Mathematics at Cornell University in 1987, and his research areas include 3-dimensional topology, gauge theory, Riemann surfaces, and computer image analysis.

**John C. Gore** John C. Gore received the Ph.D. degree in physics from the University of London, London, U.K., in 1976. He is the Director of the Institute of Imaging Science and Hertha Ramsey Cress University Professor of Radiology and Radiological Sciences, Biomedical Engineering, Physics, and Molecular Physiology and Biophysics at Vanderbilt University, Nashville, TN. His research interests include the development and application of imaging methods for understanding tissue physiology and structure, molecular imaging, and functional brain imaging. Dr. Gore is a Member of National Academy of Engineering and an elected Fellow of the American Institute of Medical and Biological Engineering, the International Society for Magnetic Resonance in Medicine (ISMRM), and the Institute of Physics (U.K.). In 2004, he was awarded the Gold Medal from the ISMRM for his contributions to the field of magnetic resonance imaging. He is Editor-in-Chief of the journal Magnetic Resonance Imaging.

**Zhaohua Ding** Zhaohua Ding received the B.E. degree in biomedical engineering from the University of Electronic Science and Technology of China, Sichuan, in 1990, the M.S. degree in computer science and the Ph.D. degree in biomedical engineering, both from The Ohio State University, Columbus, in 1997 and 1999, respectively. He was a Research Fellow at the Department of Diagnostic Radiology, Yale University, New Haven, CT, from 1999 to 2002. From July 2004, he was an Assistant Professor at the Vanderbilt University Institute of Imaging Science and Department of Radiology and Radiological Sciences. His research focuses on processing and analysis of magnetic resonance images and clinical applications.

**Anuj Srivastava** Anuj Srivastava is a Professor of Statistics at the Florida State University in Tallahassee, FL. He obtained his MS and PhD degrees in Electrical Engineering from the Washington University in St. Louis in 1993 and 1996, respectively. After spending the year 1996-97 at the Brown University as a visiting researcher, he joined FSU as an Assistant Professor in 1997. His research is focused on pattern theoretic approaches to problems in image analysis, computer vision, and signal processing. In particular, he has developed computational tools for performing statistical inferences on certain nonlinear manifolds and has published over 140 journal and conference articles in these areas.