A PATH-STRAIGHTENING METHOD FOR FINDING GEODESICS ON SHAPES SPACES OF CLOSED CURVES IN $\mathbb{R}^3$

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Abstract. In order to analyze shapes of continuous curves in $\mathbb{R}^3$, we parameterize them by arc-length and represent them as curves on a unit two-sphere. We identify the set denoting the closed curves, and study its differential geometry. To compute geodesics between any two such curves, we connect them with an arbitrary path, and then iteratively straighten this path using the gradient of an energy associated with this path. The limiting path of this path-straightening approach is a geodesic.

Next, we consider the shape space of these curves by removing shape-preserving transformations such as rotation and re-parametrization. To construct a geodesic in this shape space, we seek the shortest geodesic between the all possible transformations of the two end shapes; this is accomplished using an iterative minimization. We provide step-by-step descriptions of all the procedures, and demonstrate them with examples.

Key words. geodesic, shape analysis, 3D curves, path-straightening,

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1. Introduction. In recent years, there has been an increasing interest in analyzing shapes of objects. This research is motivated in part by the fact that shapes of objects form an important feature for characterizing them, with applications in image-based recognition, tracking, and classification of objects. For instance, shapes of boundaries of objects in images can be used to short-list possible objects present in those images. Also, shape has been used as a feature in image retrieval [5, 13, 3, 6]. Image-based shape analysis is often restricted to shapes of planar curves [19, 7, 10]; these curves can come, for example, from the boundaries of objects in 2D images. Shapes have also been used for medical diagnosis using non-invasive imaging techniques. Shapes, or growths of shapes, are often used to determine normality of anatomical parts in computational anatomy [4]. A fundamental tool, central to any differential-geometric analysis of shapes, is the construction of a geodesic path between any two given shapes in a pre-determined shape space. This tool can lead to a full statistical analysis – computation of means, covariances, tangent-space probability models – on shape spaces. As an example, the construction of geodesics and their use in statistical analysis of shapes of 2D curves is demonstrated in [18]. Extending this framework to other representations of shapes, such as 3D shapes, or shapes of surfaces in $\mathbb{R}^3$, etc, will require computational tools for constructing geodesics in appropriate shape spaces.

Although analysis of planar curves is useful in certain image understanding problems, a more general issue is to study and compare shapes of objects in 3D. Since most objects of interest are 3D objects, and 3D observations of objects using laser scans are becoming readily available, an important goal is to analyze shapes of two-dimensional surfaces in $\mathbb{R}^3$. In particular, given surfaces of two objects, the task is to quantify differences between their shapes. A differential-geometric analysis of shapes of surfaces, akin to the analysis of planar curves discussed above, remains a difficult and an unsolved problem. Therefore, several approximate methods have been pursued over the last few years. For example, the papers [15, 14] use histograms of distances on surfaces to represent and compare objects. Another approximate approach that

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has been suggested in recent years is to represent surfaces with a finite number of curves, and then compare shapes of surfaces by comparing shapes of corresponding curves [17]. These curves are typically level curves of a smooth function defined on the these surfaces. For example, as shown in Figure 1.1, we can choose a salient point on each surface, define a smooth function as a distance from that point, and study level curves of that function. Since these level curves can potentially be 3D curves, this approach requires a technique for comparing shapes of closed curves in \( \mathbb{R}^3 \). However, past research on geometric treatment of shapes of curves was restricted mainly to planar curves, and requires extension for 3D case.

In this paper, we present a differential-geometric technique for constructing geodesic paths between shapes of two arbitrary closed, continuous curves in \( \mathbb{R}^3 \). Given two curves \( p_1 \) and \( p_2 \), our basic approach is to: (i) define a shape space of all parameterized, closed curves in \( \mathbb{R}^3 \), (ii) construct an initial path connecting \( p_1 \) and \( p_2 \) in this space, and (iii) iteratively straighten this path until it becomes a geodesic path. This iteration is performed to minimize an energy associated with a path, and flows that minimize that energy are called path-straightening flows [8, 9], and more recently in [2, 12]. This methodology is quite different from the approach used in [7] where a shooting method was used to find geodesic paths between shapes. In a shooting method, one searches for a tangent direction at the first shape such that a geodesic shot in that direction reaches the target shape in a unit time. This search is based on adjusting the shooting direction in such a way that the miss function, defined as an extrinsic distance between the shape reached and the target shape, goes to zero. Intuitively, a path-straightening flow is expected to perform better than a shooting method for the following reasons:

1. While shooting, in principle, one can get stuck in a local minima of the miss function that is bounded away from zero. In other words, the resulting geodesic may not reach the target shape. In the path-straightening method, by construction, the geodesic always reaches the target shape. We admit that although both methods result in geodesics, they may not necessarily result in the shortest geodesic between the given shapes.

2. Since the shooting is performed using numerical techniques, i.e. using numerical gradient of the miss function, these iterations can become unstable if the manifold is sharply curved near the target shape. A path-straightening ap-
proach, on the other hand, is numerically more stable as it uses the gradient of path length.

3. The method proposed here can be easily extended to finding geodesics between closed curves in $\mathbb{R}^n$.

We will develop a two-step path-straightening approach to compute geodesics. In the first step, we simply consider $C$, the space of all closed curves in $\mathbb{R}^3$, and derive algorithms for computing geodesics in $C$. Here we do not take into account the shapes of these curves, and the fact that many elements of $C$ have the same shape. In the second step, we define a shape space $S$, as a quotient space of $C$, and derive algorithms for computing geodesics between elements of $S$. Here we take into account the fact that curves that differ by shape-preserving transformations have identical shape.

The rest of this paper is organized as follows. In Section 2, we present a representation of closed curves, and analyze the geometry of $C$, the space of such curves. Section 3 presents a formal discussion on the construction of path-straightening flows on $C$, followed by algorithms for computer implementations. This section also presents some illustrative examples on computing geodesic paths in $C$. Focusing on shapes of curves, Section 4 extends the construction of geodesics to the shape space $S$. The paper ends with a summary in Section 5.

2. Geometry of Shapes and Shape Spaces. In this section we introduce a geometric representation of curves that underlies our construction of geodesics and the resulting analysis of shapes.

2.1. Representations of Closed Curves. Let $p : [0, 2\pi) \mapsto \mathbb{R}^3$ be a curve of length $2\pi$, parameterized by the arc length. In this paper we will assume $p$ to be a piecewise-$C^1$ curve. For $v(s) \equiv \dot{p}(s) \in \mathbb{R}^3$, we have $\|v(s)\| = 1$ for all $s \in [0, 2\pi)$, in view of the arc-length parametrization. Here $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^3$.

The function $v$ is called the direction function of $p$ and itself can be viewed as a curve on the unit sphere $S^2$, i.e. $v : [0, 2\pi) \mapsto S^2$. Shown in Figure 2.1(a) is an illustration of this idea where a closed curve $p$ on $\mathbb{R}^3$ is represented by a curve $v$ in $S^2$. We will use the direction function $v$ to represent the curve $p$. Let $\mathcal{P}$ be the set of all such direction functions, $\mathcal{P} = \{v|v : [0, 2\pi) \mapsto S^2\}$. Since we are interested in closed curves, we establish that set as follows. Define a map $\phi : \mathcal{P} \mapsto \mathbb{R}^3$ by $\phi(v) = \int_0^{2\pi} v(s)ds$, and define

$$\mathcal{C} = \phi^{-1}(0) \equiv \{v \in \mathcal{P}|\phi(v) = 0\} \subset \mathcal{P}.$$  

Here $0$ denotes the origin in $\mathbb{R}^3$. It is easy to see that $\mathcal{C}$ is the set of all closed curves in $\mathbb{R}^3$. In the next section we will study the geometry of $\mathcal{C}$ in order to develop tools for shape analysis.

Remark: The restriction to arc-length parametrization for the curve $p$ can be relaxed by allowing for automorphisms on $[0, 2\pi)$ [11]. For any such automorphism $\phi$, the curve $p(\phi)$ is also a closed curve in $\mathbb{R}^3$. The group of such automorphisms acts on the set of closed curves and helps in matching curves, i.e. in registering points across any two given curves. Eventually, for finding geodesics between shapes, this group of automorphisms is removed along with other shape-preserving transformations.

First, we introduce some notation for studying geometry of $S^2$. Recall that geodesics on $S^2$ are great circles, and there are analytical expressions for computing them. The geodesic on $S^2$ starting at a point $x \in S^2$ in the tangent direction
\[ a \in T_x(\mathbb{S}^2) \] is defined as follows:

\[ \chi_t(x; a) = \cos(t||a||)x + \frac{\sin(t||a||)}{||a||}a. \]

\[ \chi_t \] will be used frequently in this paper to denote geodesics, or great circles, on \( \mathbb{S}^2 \). Another item that we need relates to the rotation of tangent vectors on \( \mathbb{S}^2 \). Let \( x_1 \) and \( x_2 \) be two elements in \( \mathbb{S}^2 \), and let \( a \) be a tangent to \( \mathbb{S}^2 \) at \( x \). Then, a vector defined as:

\[ \pi(a; x_1, x_2) = \begin{cases} \ a - \frac{2(a \cdot x_2)/(\|x_1 + x_2\|^2)}{\|x_1 + x_2\|}(x_1 + x_2) & \text{if } x_1 \neq -x_2 \\ \ -a & \text{if } x_1 = -x_2 \end{cases} \]

is the rotation of \( a \) to \( x_2 \) so that it is now tangent to \( \mathbb{S}^2 \) at \( x_2 \). Here, \( (a \cdot b) \) denotes the Euclidean inner product of \( a, b \in \mathbb{R}^3 \). \( \pi(\cdot, x_1, x_2) : T_{x_1}(\mathbb{S}^1) \rightarrow T_{x_2}(\mathbb{S}^1) \) is a rotation map that takes a tangent vector from \( x_1 \) to \( x_2 \); in differential geometry this is also called the parallel transport along the geodesic from \( x_1 \) to \( x_2 \).

### 2.2. Differential Geometry of \( \mathcal{C} \)

To develop a geometric framework for analyzing elements of \( \mathcal{C} \), we would like understand its tangent bundle and to impose a Riemannian structure on it. First, we focus on the set \( \mathcal{P} \). On any point \( v \in \mathcal{P} \), what form does a tangent \( f \) to \( \mathcal{P} \) take? This tangent \( f \) can be derived by constructing a one-parameter flow passing through \( v \), and by computing its velocity at \( v \). Since \( v \) is a curve on \( \mathbb{S}^2 \), the tangent \( f \) can also be viewed as a field of vectors tangent to \( \mathbb{S}^2 \) along \( v \). This idea is illustrated pictorially in Figure 2.1(b). We will interchangeably refer to \( f \) as a tangent vector on \( \mathcal{P} \) and a tangent vector field on points along \( v \in \mathbb{S}^2 \). The space of all such tangent vectors, denoted by \( T_v(\mathcal{P}) \), is given by:

\[ T_v(\mathcal{P}) = \{ f|f : [0, 2\pi) \rightarrow \mathbb{R}^3, \forall s, (f(s) \cdot v(s)) = 0 \} \]

\( f(s) \) and \( v(s) \) are vectors in \( \mathbb{R}^3 \). Let \( f \in T_v(\mathcal{P}) \) be a vector field on \( v \) such that it is also tangent to \( \mathcal{C} \). It can be shown that \( f \) satisfies \( \int_0^{2\pi} f(s)ds = 0 \). That is,

\[ T_v(\mathcal{C}) = \{ f|f : [0, 2\pi) \rightarrow \mathbb{R}^3, (f(s) \cdot v(s)) = 0, \int_0^{2\pi} f(s)ds = 0 \forall s \} \]

To see that, let \( \alpha(t) \) be a path in \( \mathcal{C} \) such that \( \alpha(0) = v \). Since \( \alpha(t) \in \mathcal{C} \), we have \( \int_0^{2\pi} \alpha(t)(s)ds = 0 \), for all \( t \). Taking the derivative with respect to \( t \) and setting \( t = 0 \), we get \( \int_0^{2\pi} \dot{\alpha}(t)(s)ds = 0 \). For every tangent vector \( f \) at \( v \) there is a corresponding

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**Fig. 2.1.** (a): A closed curve in \( \mathbb{R}^3 \) is denoted by a curve on \( \mathbb{S}^2 \). (b): For a curve \( v \) on \( \mathbb{S}^2 \), \( f \) is vector field to \( \mathbb{S}^2 \) on \( v \).
flow \( \alpha \), such that \( f = \dot{\alpha}(0) \), and therefore, this property is satisfied by all tangent vectors.

**Riemannian Structure:** To impose a Riemannian structure on \( \mathcal{P} \), we will assume the following inner product on \( T_v(\mathcal{P}) \): for \( f, g \in T_v(\mathcal{P}) \), \( \langle f, g \rangle = \int_0^{2\pi} (f(s) \cdot g(s)) ds \).

Consider the linear mapping \( d\phi_v : T_v(\mathcal{P}) \to \mathbb{R}^3 \) defined by \( d\phi_v(f) = \int_0^{2\pi} f(s) ds \). Similar to the argument in [12], it can be shown that \( d\phi_v \) is surjective, as long as \( v([0,2\pi]) \) is not contained in a one-dimensional subspace of \( \mathbb{R}^3 \), and therefore \( \mathcal{C} \) is a co-dimension three submanifold of \( \mathcal{P} \). The adjoint of \( d\phi_v \), \( d\phi_v^* : \mathbb{R}^3 \to T_v(\mathcal{P}) \) is the unique linear transformation with the property that for all \( f \in T_v(\mathcal{P}) \) and \( x \in \mathbb{R}^3 \), \( \langle d\phi_v(f) \cdot x \rangle = \langle f, d\phi_v^*(x) \rangle \). This adjoint is given by \( d\phi_v^*(x) \equiv f \) such that \( f(s) = x - (x \cdot v(s)) v(s) \). In other words, \( d\phi_v^* \) takes a vector \( x \in \mathbb{R}^3 \) and forms a tangent vector-field on \( v \) by making \( x \) perpendicular to \( v(s) \) for all \( s \) (or by projecting \( x \) onto the tangent space \( T_{v(s)}(\mathbb{S}^2) \) for each \( s \)). This formula makes explicit the role of \( v \) in definition of \( d\phi_v^* \).

**Proposition 2.1.** The range space of \( d\phi_v^* \) is the orthogonal complement of the null space of \( d\phi_v \):

\[
\{ f \in T_v(\mathcal{P}) | f = d\phi_v^*(x) \text{ for some } x \in \mathbb{R}^3 \} = \{ f \in T_v(\mathcal{P}) | d\phi_v(f) = 0 \}^\perp.
\]

**Proof:**

If \( f \) is in range space of \( d\phi_v^* \), i.e. there exists a \( x \) such that \( d\phi_v^*(x) = f \). Then, for any \( g \) in the null space of \( d\phi_v \), we have \( \langle f, g \rangle = \langle d\phi_v^*(x), g \rangle = (x \cdot d\phi_v(g)) = 0 \). That is, \( f \) is perpendicular to the null space of \( \phi_v \). The proposition follows from the fact that the dimension of range space of \( d\phi_v^* \) is equal to the co-dimension of the null space of \( d\phi_v \).

Q.E.D.

Let \( e_1, e_2, \) and \( e_3 \) be the canonical basis of \( \mathbb{R}^3 \), and define \( E_{v,i} = d\phi_v^*(e_i) \in T_v(\mathcal{P}) \) for \( i = 1, 2, 3 \). The functions \( E_{v,i}, i = 1, 2, 3 \) form a basis of the range space of \( d\phi_v^* \). Using Eqn. 2.5, these three functions also form a basis for the orthogonal complement space of \( d\phi_v \).

With this framework, we develop tools for projecting a curve \( v \in \mathcal{P} \) into \( \mathcal{C} \). Also, we derive a mechanism for projecting \( f \in T_v(\mathcal{P}) \) into \( T_v(\mathcal{C}) \).

1. **Projection of \( v \) into \( \mathcal{C} \):** If \( v \notin \mathcal{C} \), i.e. it denotes an open curve, we want to close it by projecting it into \( \mathcal{C} \). We perform this iteratively by moving perpendicular to the level sets of the map \( \phi \) in such a way that \( \phi \) of the resulting point goes towards origin in \( \mathbb{R}^3 \). In other words, we seek a direction \( g \) in \( T_v(\mathcal{P}) \) such that a perturbation of \( v \) in that direction results in \( \phi(v+g) = 0 \), up to first order. Since \( g \) is perpendicular to the level set of \( \phi \) at \( v \), it is orthogonal to the null space of \( d\phi_v \), and by Eqn. 2.5, it is in the range space of \( d\phi_v^* \). Therefore, \( g \) can be written as a linear combination of the functions \( E_{i,v}, i = 1, 2, 3 \). Using Taylor’s approximation, we get

\[
\phi(v+g) \approx \phi(v) + d\phi_v(g) = 0 \quad \Rightarrow \quad g = -A_v^{-1}\phi(v),
\]

where \( A_v \in \mathbb{R}^{3 \times 3} \) is the Jacobian of \( d\phi_v \) restricted to the space orthogonal to the null space of \( d\phi_v \). The Jacobian \( A_v \) can also be specified using the same basis functions. An algorithm to project any \( v \in \mathcal{P} \) onto \( \mathcal{C} \) is given next.


**Fig. 2.2.** Projection of open curves into $\mathcal{C}$ using Subroutine 1. The plots show open curves (plane lines) and their projections (marked lines) in $\mathbb{R}^3$.

**SUBROUTINE 1 (Projection into $\mathcal{C}$).**

while $\text{norm}(\phi(v)) > \epsilon$

for $i = 1 : 3$

$$E_{i,v} = d\phi^*_v(e_i)$$

$$A_v(:, i) = d\phi_v(E_{i,v})$$

end

$b = -A_{-1}^v\phi(v)$

$g = b(1)E_{1,v} + b(2)E_{2,v} + b(3)E_{3,v}$

for all $s$ in $[0, 2\pi)$

$v(:, s) = \chi_1(v(s); g(s))$  \(\chi\) is defined in Eqn. 2.2

end $s$

end

Shown in Figure 2.2 are three examples of this projection. In the top row, we show three open curves in $\mathcal{P}$ (drawn in plane lines) and their projections into $\mathcal{C}$ (drawn in marked lines). The bottom row shows the corresponding curves on $S^2$.

2. **Projecting $f \in T_v(\mathcal{P})$ into the subspace $T_v(\mathcal{C})$.** We will assume that $v \in \mathcal{C}$. For a vector field $f \in T_v(\mathcal{P})$, we have that $(f(s) \cdot v(s)) = 0$. However, $\int_0^{2\pi} f(s)ds$ may not be zero, and we need to ensure that for it to be an element of $T_v(\mathcal{C})$. This can be achieved using a one-time projection:

**SUBROUTINE 2 (Projection from $T_v(\mathcal{P})$ into $T_v(\mathcal{C})$).**

for $i = 1 : 3$

$$E_{i,v} = d\phi^*_v(e_i)$$

$$A_v(:, i) = d\phi_v(E_{i,v})$$

end

$b = -A_{-1}^v\phi(v)$

$g = b(1)E_{1,v} + b(2)E_{2,v} + b(3)E_{3,v}$

for all $s$ in $[0, 2\pi)$

$f(:, s) = f(s) - g(s)$

end $s$

3. **Path-Straightening Flows in $\mathcal{C}$.** Now we present our approach for constructing geodesic flows on $\mathcal{C}$. This approach is based on the use of path-straightening flows. That is, we connect the two given curves by an arbitrary path in $\mathcal{C}$, and then iteratively straighten it, or shorten it, using a gradient approach until we reach a fixed
3.1. Formal Ideas. For any two closed curves, denoted by \( v_0 \) and \( v_1 \) in \( \mathcal{C} \), we are interested in finding a geodesic path between them in \( \mathcal{C} \). Our approach is to start with any path \( \alpha(t) \) connecting \( v_0 \) and \( v_1 \). That is \( \alpha : [0, 1] \mapsto \mathcal{C} \) such that \( \alpha(0) = v_0 \) and \( \alpha(1) = v_1 \). (A pictorial example of a path \( \alpha \) on \( \mathcal{C} \) is shown in Figure 3.1(a)). Then, we iteratively “straighten” \( \alpha \) until it achieves a local minimum of the energy:

\[
E(\alpha) \equiv \frac{1}{2} \int_0^1 \left( \frac{d\alpha}{dt}(t), \frac{d\alpha}{dt}(t) \right) dt ,
\]

over all paths from \( v_0 \) to \( v_1 \). It will be shown later that a critical point of \( E \) is a geodesic on \( \mathcal{C} \). However, it is possible that there are multiple geodesics between a given pair \( v_0 \) and \( v_1 \), and a local minimum of \( E \) may not correspond to the shortest of all geodesics. Therefore, this approach has the limitation that it finds a geodesic between a given pair but may not reach the shortest geodesic. One can use certain stochastic techniques to increase the probability of reaching the shortest geodesic but these are not explored in this paper.

Let \( \mathcal{H} \) be the set of all paths in \( \mathcal{C} \), parameterized by \( t \in [0, 1] \), and \( \mathcal{H}_0 \) be the subset of \( \mathcal{H} \) of paths that start at \( v_0 \) and end at \( v_1 \). The tangent spaces of \( \mathcal{H} \) and \( \mathcal{H}_0 \) are:

\[
T_{\alpha}(\mathcal{H}) = \{ w | \forall t \in [0, 1], w(t) \in T_{\alpha(t)}(\mathcal{C}) \} ,
\]

where \( T_{\alpha(t)}(\mathcal{C}) \) is as specified in Eqn. 2.4, and

\[
T_{\alpha}(\mathcal{H}_0) = \{ w \in T_{\alpha}(\mathcal{H}) | w(0) = w(1) = 0 \} .
\]

To understand this space, consider a path \( \alpha \in \mathcal{H}_0 \) and an element \( w \in T_{\alpha}(\mathcal{H}_0) \). Recall that for any \( t \), \( \alpha(t) \) is also a curve on \( S^2 \), which in turn corresponds to a closed curve in \( \mathbb{R}^3 \). Now, \( w \) is path of vector fields such that for any \( t \in [0, 1] \), \( w(t) \) is a tangent vector field along the curve \( \alpha(t) \) on \( S^2 \). That is, \( w(t)(s) \) is a vector tangent to \( S^2 \) at the point \( \alpha(t)(s) \). Furthermore, \( \int_0^{2\pi} w(t)(s) ds = 0 \) for all \( t \in [0, 1] \). Our study of paths on \( \mathcal{H} \) requires the use of covariant derivatives and integrals of vector fields along these paths.
Definition 3.1 (Covariant Derivative, [1](pg. 309)). For a given path \( \alpha \in \mathcal{H} \) and a vector field \( w \in T_\alpha(\mathcal{H}) \), one defines the covariant derivative of \( w \) along \( \alpha \) to be the vector field obtained by projecting \( \frac{Dw}{dt}(t) \) onto the tangent space \( T_{\alpha(t)}(\mathcal{C}) \), for all \( t \). It is denoted by \( \frac{Dw}{dt} \). Similarly, a vector field \( u \in T_\alpha(\mathcal{H}) \) is called a covariant integral of \( w \) along \( \alpha \) if the covariant derivative of \( u \) is \( w \), i.e. \( \frac{Du}{dt} = w \).

To make \( \mathcal{H} \) a Riemannian manifold, we use the Palais metric [16]: for \( w_1, w_2 \in T_\alpha(\mathcal{H}) \),

\[
\langle \langle w_1, w_2 \rangle \rangle = \langle w_1(0), w_2(0) \rangle + \int_0^1 \left\langle \frac{Dw_1}{dt}(t), \frac{Dw_2}{dt}(t) \right\rangle dt,
\]

where \( \frac{Dw}{dt} \) denotes the vector field along \( \alpha \) which is the covariant derivative of \( w \). With respect to the Palais metric, \( T_\alpha(\mathcal{H}_0) \) is a closed linear subspace of \( T_\alpha(\mathcal{H}) \), and \( \mathcal{H}_0 \) is a closed subspace of \( \mathcal{H} \).

Our goal is to find the minimizer of \( E \) in \( \mathcal{H}_0 \), and we will use a gradient flow to minimize \( E \). Therefore, we wish to find the gradient of \( E \) in \( T_\alpha(\mathcal{H}_0) \). To do this, we first find the gradient of \( E \) in \( T_\alpha(\mathcal{H}) \) and then project it into \( T_\alpha(\mathcal{H}_0) \).

Theorem 3.2. The gradient vector of \( E \) in \( T_\alpha(\mathcal{H}) \) is given by a vector field \( q \) such that \( Dq/dt = d\alpha/dt \) and \( q(0) = 0 \). In other words, \( q \) is the covariant integral of \( d\alpha/dt \) with zero initial value at \( t = 0 \).

Proof:

Define a variation of \( \alpha \) to be a smooth function \( h : [0,1] \times (-\epsilon, \epsilon) \rightarrow \mathcal{H} \) such that \( h(t,0) = \alpha(t) \) for all \( t \in [0,1] \). The variational vector field corresponding to \( h \) is given by \( v(t) = h\tau(t,0) \) where \( \tau \) denotes the second argument in \( h \). Thinking of \( h \) as a path of curves in \( \mathcal{H} \), we define \( E(\tau) \) as the energy of the curve obtained by restricting \( h \) to \([0,1] \times \{ \tau \} \). That is,

\[
E(\tau) = \frac{1}{2} \int_0^1 \langle h_t(t,\tau), h_t(t,\tau) \rangle dt.
\]

We now compute,

\[
E'(0) = \int_0^1 \left\langle \frac{Dh}{d\tau}(t,0), h_t(t,0) \right\rangle dt = \int_0^1 \left\langle \frac{Dh}{d\tau}(t,0), h_t(t,0) \right\rangle dt = \int_0^1 \left\langle \frac{Dv}{dt}(t), \frac{d\alpha}{dt}(t) \right\rangle dt,
\]

since \( h_t(t,0) \) is simply \( d\alpha/dt(t) \). Now, the gradient of \( E \) should be a vector field \( q \) along \( \alpha \) such that \( E'(0) = \langle (v,q) \rangle \). That is,

\[
E'(0) = \langle (v(0), q(0)) \rangle + \int_0^1 \left\langle \frac{Dv}{dt} \times \frac{Dq}{dt} \right\rangle dt.
\]

From this expression it is clear that \( q \) must satisfy the initial condition \( q(0) = 0 \) and the ordinary (covariant) differential equation \( \frac{Dq}{dt} = \frac{d\alpha}{dt} \). Q.E.D.

Given \( d\alpha/dt \), the vector field \( q \) is obtained using numerical techniques for covariant integration, as described in the next section. Next, we want to project a tangent field \( q \in T_\alpha(\mathcal{H}_0) \) to the space \( T_\alpha(\mathcal{H}_0) \).

Definition 3.3 (Covariantly Constant). A vector field \( w \) along the path \( \alpha \) is called covariantly constant if \( Dw/dt \) is zero at all points along \( \alpha \).

Definition 3.4 (Geodesic). A path is called a geodesic if its velocity vector field is covariantly constant. That is, \( \alpha \) is a geodesic if \( \frac{Dw}{dt}(\frac{d\alpha}{dt}) = 0 \) for all \( t \).
DEFINITION 3.5 (Covariantly Linear). A vector field $w$ along the path $\alpha$ is called covariantly linear if $Dw/dt$ is a covariantly constant vector field.

LEMMA 3.6. The orthogonal complement of $T_\alpha(H_0)$ in $T_\alpha(H)$ is the space of all covariantly linear vector fields $w$ along $\alpha$.

Proof: Suppose $v \in T_\alpha(H_0)$ (i.e. $v(0) = v(1) = 0$), and $w \in T_\alpha(H)$ is covariantly linear. Then, using (covariant) integration by parts:

$$\langle v, w \rangle = \int_0^1 \left\langle \frac{Dv(t)}{dt}, \frac{Dw(t)}{dt} \right\rangle dt = \int_0^1 \left\langle v, \frac{Dw(t)}{dt} \right\rangle dt - \int_0^1 \left\langle v, \frac{D}{dt} \left( \frac{Dw(t)}{dt} \right) \right\rangle dt = 0.$$

Hence, $T_\alpha(H_0)$ is orthogonal to the space of covariantly linear vector fields along $\alpha$ in $T_\alpha(H)$. This proves the space of covariantly linear vector fields is contained in the orthogonal complement of $T_\alpha(H_0)$. To prove that these two spaces are equal, observe first that given any choice of tangent vectors at $\alpha(0)$ and $\alpha(1)$, there is a unique covariantly linear vector field interpolating them. It follows that every vector field along $\alpha$ can be uniquely express as the sum of a covariantly linear vector field and a vector field in $T_\alpha(H_0)$. The lemma then follows.

DEFINITION 3.7 (Parallel Translation). A vector field $u$ is called the forward parallel translation of a tangent vector $w \in T_{\alpha(0)}(C)$, along $\alpha$, if and only if $w(0) = u$ and $\frac{Du(t)}{dt} = 0$ for all $t \in [0, 1]$.

Similarly, $u$ is called the backward parallel translation of a tangent vector $w \in T_{\alpha(1)}(C)$, along $\alpha$, when for $\tilde{\alpha}(t) \equiv \alpha(1 - t)$, $u$ is the forward parallel translation of $w$ along $\tilde{\alpha}$. It must be noted that parallel translations, forward or backward, lead to vector fields that are covariantly constant.

According to Lemma 1, to project the gradient $q$ into $T_\alpha(H_0)$, we simply need to subtract off a covariantly linear vector field which agrees with $q$ at $t = 0$ and $t = 1$. Clearly, the correct covariantly linear field is simply $\tilde{q}(t)$, where $\tilde{q}(t)$ is the covariantly constant field obtained by parallel translating $q(1)$ backwards along $\alpha$. Hence, we have proved the following theorem.

THEOREM 3.8. Let $\alpha : [0, 1] \mapsto C$ be a path, $\alpha \in H_0$. Then, with respect to the Palais metric:

1. The gradient of the energy function $E$ on $H$ at $\alpha$ is the vector field $q$ along $\alpha$ satisfying $q(0) = 0$ and $\frac{Dq}{dt} = \frac{dq}{dt}$.

2. The gradient of the energy function $E$ restricted to $H_0$ is $w(t) = q(t) - \tilde{q}(t)$, where $q$ is the vector field defined in the previous item, and $\tilde{q}$ is the vector field obtained by parallel translating $q(1)$ backwards along $\alpha$.

To understand this result, consider a simpler case of path-straightening flow in a Euclidean space. That is, we are given a path $\alpha(t)$ in $\mathbb{R}^n$ that we want to straighten into a geodesic (a straight line) joining $\alpha(0)$ and $\alpha(1)$. As shown in Figure 3.1(b), the particularization of Theorem 3.8 leads to simple, intuitive results. In $\mathbb{R}^n$, the covariant derivative (integral) is replaced by the ordinary derivative (integral). Therefore, $q(t) = \int_0^t \frac{d\alpha}{dt}(s) ds = \alpha(t) - \alpha(0)$, and since $q(1) = \alpha(1) - \alpha(0)$, $\tilde{q}(t)$ is simply $\alpha(1) - \alpha(t)$. For any point $\alpha(t)$ on the curve, the gradient vector is given by $w(t) = (\alpha(t) - \alpha(0)) - t(\alpha(1) - \alpha(0))$. As shown in Figure 3.1(b), it is the vector that takes the point $\alpha(t)$ to the corresponding point on the straight line (the geodesic), and that seems a natural choice to straighten $\alpha$ into a straight line.
Theorem 3.9. For a given pair $v_0, v_1 \in C$, a critical point of $E$ on $H_0$ is a geodesic on $C$ connecting $v_0$ and $v_1$.

Proof: Let $\alpha$ be a critical point of $E$ on $H_0$. That is, the gradient of $E$ is zero along $\alpha$. Since the gradient vector field is given by $q(t) - t\tilde{q}(t)$, we have that $q(t) = t\tilde{q}(t)$ for all $t$. Therefore,

$$\frac{d\alpha}{dt} = Dq = D(\tilde{q}) = \tilde{q}.$$  

Since $\tilde{q}$ is a parallel translation of $q(1)$, it is covariantly constant, and therefore, the velocity field $\frac{d\alpha}{dt}$ is covariantly constant. By definition, this implies that $\alpha$ is a geodesic.

Q.E.D.

3.2. Computer Implementations. In this section, we provide step-by-step details for different procedures mentioned in the last section. In particular, we provide algorithms for: (i) finding the direction function $v$ of a given closed curve $p$, (ii) given any two closed curves, $v_0$ and $v_1$, initializing a path $\alpha$ connecting them in $C$, (iii) computing the velocity vector $\frac{d\alpha}{dt}$ for a given path $\alpha$, (iv) computing the covariant integral $q$ of $\frac{d\alpha}{dt}$, (v) computing the backward parallel transport $\tilde{q}$ of $q(1)$, and (vi) updating the path $\alpha$ along the gradient direction given by the vector field $w$.

We explain these procedures one by one next.

1. Direction Function Representation of closed curves: The first computational step in our analysis is to find an element of $C$ for a given 3D curve. This curve is often specified in form of a finite collection of ordered, non-uniformly spaced points in $\mathbb{R}^3$. We first obtain a re-sampling of these points in a uniform fashion and then compute the tangent vectors. The re-sampling is achieved using an iterative procedure as follows:

Subroutine 3 (Uniform Re-sampling of Curve).

1. Set $x_{m+1} = x_1$.
2. Compute $\rho_i = \|x_{i+1} - x_i\|$, $i = 1, \ldots, m$.
3. While standard-deviation($\{\rho_i\}$) > $\epsilon$:
   1. $s_i = \sum_{j=1}^{i} \rho_j$, $i = 1, \ldots, m$.
   2. $t = \left(\frac{1}{n}\right)s_m$.
   3. $k_j = \text{argmin}_i (s_i \geq t_j)$, $j = 1, \ldots, n$.
   4. $y_1 = x_1$.
   5. For $j = 1, \ldots, n-1$:
      1. $y_{j+1} = \frac{((t_j - x_{k_j-1})x_{k_j+1} + (x_{k_j} - t_j)x_{k_j})/\|w_j\|}{\|w_j\|}$, $\rho_j = \|w_j\|$.
   6. End for.
   7. Set $m = n$ and $x = y$.
4. End while.

Note that $s_m$ is the total length of the piecewise-linear curve formed by the given $x_i$s, and we are simply re-sampling this piecewise-linear curve at spacings of $s_m/n$ along its arc-length. In view of this piecewise-linear assumption, the resulting $y_j$s may not be uniformly spaced but they are more uniform than
the original set of \( x_i \)s. A repeated application of this idea results in a uniformly sampled set. Shown in Figure 3.2 is an example. The given curve with \( m = 200 \) is shown in the left panel; it is re-sampled repeatedly for \( n = 30 \) with results shown in next two panels. To show that points become increasingly uniform, we show the standard deviation of \( \rho_i \)s at every iteration. A standard deviation of zero implies that the points are uniformly spaced.

2. **Initialize the path** \( \alpha \): Given \( v_0 \) and \( v_1 \) in \( \mathcal{C} \), we want to form a path \( \alpha : [0, 1] \rightarrow \mathcal{C} \) such that \( \alpha(0) = v_0 \) and \( \alpha(1) = v_1 \). There are several ways of doing this. One is to form 3D coordinates \( p_0 \) and \( p_1 \), respectively, associated with the two shapes, and connect \( p_0(s) \) and \( p_1(s) \) linearly, for all \( s \), using 
\[
p_t(s) = tp_1(s) + (1 - t)p_0(s).
\]
The intermediate curves are not uniformly sampled although they will be closed. We can use Subroutine 3 to re-sample them uniformly. The other idea is to use the fact that \( v_0(s), v_1(s) \in \mathbb{S}^2 \), and construct a path in \( \mathbb{S}^2 \) from one point to another, parameterized by \( t \). We summarize this idea in the following subroutine.

**SUBROUTINE 4** (Initialize a path \( \alpha \)).

\[
\text{for all } s \in [0, 2\pi] \\
\quad \text{define } \theta(s) = \cos^{-1}(v_0(s) \cdot v_1(s)) \\
\quad \text{define } f(s) = v_1(s) - (v_0(s) \cdot v_1(s))v_0(s), \text{ and } f(s) = \theta(s)/\|f(s)\|. \\
\text{end } s
\]

\[
\text{for all } t \in [0, 1] \\
\quad \text{for all } s \in [0, 2\pi] \\
\quad \quad \text{define } \alpha(t)(s) = \chi_t(v_0(s); f(s)) \\
\quad \text{end } s \\
\quad \text{project } \alpha(t) \text{ into } \mathcal{C} \text{ using Subroutine 1.} \\
\text{end } t
\]

In case \( v_0(s) \) and \( v_1(s) \) are antipodal points on \( \mathbb{S}^2 \), and thus \( f(s) = 0 \), one can arbitrarily choose a path connecting them on the sphere. However, this situation occurs in practice with probability zero and is not an practical issue.

3. **Vector Field** \( d\alpha/dt \): In order to compute the gradient of \( E \) in \( T_\alpha(H) \), we first need to compute the path velocity \( d\alpha/dt \). For a continuous path, \( d\alpha/dt(t) \) automatically lies in \( T_{\alpha(t)}(\mathcal{C}) \), but in the discrete case one has to ensure this property using additional steps. This process uses the approximation \( x'(t) \approx (x(t) - x(t - \epsilon))/\epsilon \), modified to account for the nonlinearity of \( \mathcal{C} \). Let the interval \([0, 1]\) be divided into \( k \) uniform bins. The procedure for computing
\( \frac{d\alpha}{dt} \) at these discrete times is summarized next.

**Subroutine 5 (Computation of \( \frac{d\alpha}{dt} \) along \( \alpha \)).**

\[
\text{for } \tau = 1, \ldots, k \\
\text{for all } s \in [0, 2\pi] \\
\theta(s) = k \cos^{-1}(\alpha(\tilde{\tau}(s)) \cdot \alpha(\tilde{\tau}(s+1)) \cdot \alpha(\tilde{\tau}(s))) \\
f(s) = -\alpha(\tilde{\tau}(s+1)) + (\alpha(\tilde{\tau}(s+1)) \cdot \alpha(\tilde{\tau}(s+1)) \cdot \alpha(\tilde{\tau}(s))) \\
\frac{d\alpha}{dt}(\tilde{\tau}(s)) = \theta(s)f(s)/\|f(s)\|. \\
\text{end } s \\
\text{project } \frac{d\tau}{dt}(\tilde{\tau}(s)) \text{ into } T_{\alpha(\tilde{\tau})}(\mathcal{C}) \text{ using Subroutine 2.} \\
\text{end } \tau.
\]

Now we have a vector field \( \frac{d\alpha}{dt} \in T_{\alpha}(\mathcal{H}) \) along a given path \( \alpha \in \mathcal{H} \).

**4. Computation of Vector field \( q \):** We seek a vector field \( q \) such that \( q(0) = 0 \) and \( \frac{Dq}{dt} = \frac{d\alpha}{dt} \). In other words, \( q \) is the covariant integral of the vector field \( \frac{d\alpha}{dt} \).

**Subroutine 6 (Covariance Integration of \( \frac{d\alpha}{dt} \) to form \( q \)).**

\[
\text{for } \tau = 0, 1, 2, \ldots, k - 1, \\
\text{for all } s \in [0, 2\pi] \\
define \|q(\tilde{\tau}(s))\(q(\tilde{\tau}(s)), \alpha(\tilde{\tau}(s)), \alpha(\tilde{\tau}(s+1))\). \\
(\pi \text{ is defined in Eqn. 2.3}) \\
set q(\tilde{\tau}(s+1)) = \frac{1}{k} \frac{d\tau}{dt}(\tilde{\tau}(s+1)) + q(\tilde{\tau}(s)). \\
\text{end } s \\
\text{end } \tau.
\]

\( q(\tilde{\tau}(s)) \) is the parallel transport of \( q(\tilde{\tau}(s)) \) from \( T_{\alpha(\tilde{\tau})}(\mathcal{C}) \) to \( T_{\alpha(\tilde{\tau}(s+1))}(\mathcal{C}) \). This subroutine results in the gradient vector field \( \{q(\tilde{\tau}(s)) \in T_{\alpha(\tilde{\tau})}(\mathcal{C}) \mid \tau = 1, \ldots, k \} \).

**5. Covariant Vector Field \( \tilde{q} \):** Given \( q(1) \), we need to find a vector field \( \tilde{q} \) along the path \( \alpha \in \mathcal{C} \) that is the backward parallel transport of \( q(1) \). We have already computed the points \( \alpha(0), \alpha(1/k), \alpha(2/k), \ldots, \alpha(1) \). Each \( \alpha(\tilde{\tau}(s)) \) is an element of \( \mathcal{C} \), i.e. it is a curve on \( S^2 \). We will perform the backward parallel transport iteratively, as follows.

**Subroutine 7 (Backward Parallel Transport).**

\[
\text{set } \tilde{q}(1) = q(1) \\
\text{let } l = (q(1), q(1))^{1/2} \\
\text{for } \tau = k - 1, k - 2, \ldots, 2, 1, 0 \\
\text{for all } s \in [0, 2\pi] \\
q(\tilde{\tau}(s)) = \pi(\tilde{q}(\tilde{\tau}(s+1)), \alpha(\tilde{\tau}(s)), \alpha(\til{\tau}(s))) \\
\text{end } s \\
\text{project } \tilde{q}(\til{\tau}(s)) \text{ into } T_{\alpha(\til{\tau})}(\mathcal{C}) \text{ using Subroutine 2.} \\
\text{let } l_1 = (\langle \til{q}(\til{\tau}(s)), \til{q}(\til{\tau}(s)) \rangle)^{1/2} \\
\text{set } \tilde{q}(\til{\tau}(s)) = \til{q}(\til{\tau}(s))/l_1; \\
\text{end } \tau.
\]

**6. Gradient of \( E \):** With the computation of \( q \) and \( \tilde{q} \) along the path \( \alpha \), the gradient vector field of \( E \) is given by: for any \( \tau \in \{0, 1, \ldots, k\} \) and \( s \in [0, 2\pi] \)

\[
(3.2) \quad w(\til{\tau}_k)(s) = q(\til{\tau}_k)(s) - (\til{\tau}_k)\til{q}(\til{\tau}_k)(s) \in T_{\alpha(\til{\tau}_k)(s)}(S^2).
\]

**7. Update in Gradient Direction:** Now that we have computed the gradient vector field \( w \) on the current path \( \alpha \), we update this path in the direction
given by $w$: for $\tau = 1, 2, \ldots, k$ and $s \in [0, 2\pi)$,

\begin{equation}
\alpha(t)(s) = \chi_1(\alpha(t)(s); w(t)(s)).
\end{equation}

### 3.3. Algorithm to Compute Geodesic in $C$.
Here we summarize our algorithm to compute a geodesic path between any two given closed curves in $\mathbb{R}^3$. We assume that the curves are available in form of sampled points on these curves.

**Algorithm 1** (Find a geodesic between two curves in $C$).

1. Compute the representations of each curve in $C$ using Subroutine 3. Denote these elements by $v_0$ and $v_1$, respectively.
2. Initialize a path $\alpha$ between $v_0$ and $v_1$ using Subroutine 4.
3. Compute the velocity vector field $d\alpha/dt$ along the path $\alpha$ using Subroutine 5.
4. Compute the covariant integral of $d\alpha/dt$, denoted by $q$, using Subroutine 6.
   If $\sum_{\tau=1}^{k} \langle d\alpha/dt(\tau), d\alpha/dt(\tau) \rangle$ is small, then stop. Else, continue to the next step.
5. Compute the backward parallel transport of the vector $q(1)$ along $\alpha$ using the Subroutine 7.
6. Compute the full gradient vector field of the energy $E$ along the path $\alpha$, denoted by $w$, using Eqn. 3.2.
7. Update $\alpha$ using Eqn. 3.3. Return to Step 3.

The desired geodesic path is given by the resulting $\alpha$, and its length is given by $d_C(v_0, v_1) = \left( \langle d\alpha/dt(0), d\alpha/dt(0) \rangle \right)^{1/2}$. For a later use, we highlight $d\alpha/dt(0)$ as the initial velocity vector in $T_{\alpha(0)}(C)$ that generates the geodesic at $\alpha(0)$.

### 3.4. Experimental Results.
In this section we describe some computer experiments for generating geodesic paths between shapes in $C$.

Let the two curves of interest be:

\[ p_0(t) = (a \cos(t), b \sin(t), \sqrt{b^2 - a^2 \sin^2(t)}), \quad p_1(t) = (a(1 + \cos(t)), \sin(t), 2 \sin(t/2)) \]

and we want to compute a geodesic path between them in $C$. Shown in Figure 3.3 are the results. The panels in (a) show the two curves. The first curve is an example of a *bicylinder* and the second one is an example of a *Viviani* curve. We apply Algorithm 1 on these two curves to generate a geodesic path between them. Panel (b) shows the evolution of the energy $E$ during the iterations in Algorithm 1. Panels (c) show two views of the geodesic path drawn on $S^2$; the red curve denotes $v_0$, the green curve denotes $v_1$, and the blue curves denote intermediate points on the resulting geodesic $\alpha$. The bottom row shows two views of the geodesic path as closed curves in $\mathbb{R}^3$.

Shown in Figure 3.4 is another example, where the two end shapes (top left two panels), evolution of the energy (top right), and two views of the final geodesic path (middle and lower panels) are displayed.

### 4. Geodesics on Shape Space.
So far we have consider the space of closed curves $C$ and have presented a numerical technique for finding geodesics between elements of $C$. However, when considering shapes of these curves we observe that many points in $C$ can represent the same shape. This is because a rigid rotation or a re-parametrization can change the representation of a curve but not its shape. To study shapes of curves, and not the curves themselves, we have to remove these shape-preserving transformations from the representation. Towards that end, we introduce the shape space as a quotient space of $C$, modulo shape preserving transformations.
such as rigid rotations and re-parameterizations. Then, we describe techniques for computing geodesic paths on the shape space.

4.1. Shape Space. The main goal of this paper is to analyze shapes of closed curves in $\mathbb{R}^3$, and we seek a way to represent shapes of these curves. By representing a closed curve $p$ with its direction function $v \in \mathcal{C}$, we have already removed the translation variability. This is due to the fact that a rigid translation of a closed curve does not change its direction function, and hence its representation. Similarly, by requiring the curves to be of length $2\pi$, we have removed the scale variability also. On the other hand, a rigid rotation or a re-parametrization do change the velocity field along the curve, so we have to remove them explicitly, as described next.

For the unit circle $S^1$, and an angle $\theta \in S^1$, the re-parametrization of a curve $v \in \mathcal{C}$, defined by:

$$ (\theta \circ v)(s) = v((s - \theta) \mod 2\pi) $$

(4.1)

It is easy to see that $\theta \circ v$ is also in $\mathcal{C}$, and it denotes a curve with the same shape as that of $v$. $S^1$ is called the re-parametrization group. Similarly for rotation: consider the space of $3 \times 3$ rotation matrices, the special orthogonal group $SO(3)$, and for $O \in SO(3)$ we define a rotated shape

$$ Ov(s) = O \cdot v(s), \quad \forall s \in [0, 2\pi) $$

(4.2)

$SO(3)$ is called the rotation group. Both these groups, $S^1$ and $SO(3)$, act on $\mathcal{C}$ (see [1] for definition of group action), and they commute with each other. We can define the orbit associated with a curve $v \in \mathcal{C}$ as:

$$ \mathcal{C}_v = \{ w \in \mathcal{C} | w = \theta \circ Ov, \quad \forall \theta \in S^1, \quad \forall O \in SO(3) \} $$

(4.3)
All the elements of \( C_v \) has the same shape, and one can view memberships of these orbits as an equivalence relation. These orbits divide \( C \) into disjoint equivalence classes, each subset being associated with a unique shape. Hence, the shape space is defined to be the quotient space \( S = C/(S^1 \times SO(3)) \). It must be noted that the action of \( S^1 \times SO(3) \) on \( C \) is isometric, i.e. for any \( g, h \in T_v(C) \), we have \( \langle g, h \rangle = \langle \theta \circ O g, \theta \circ O h \rangle \), for any \( O \in SO(3), \theta \in S^1 \). Therefore, the geodesic distance between any two points in \( C \) does not change if we re-parameterize and rotate both of them in the same way, i.e. \( d_C(c_v, c_{v1}) = d_C(\theta \circ Ov_0, \theta \circ Ov_1) \), \( \forall \theta \in S^1, \forall O \in SO(3) \).

4.2. Geodesics in \( S \). The shape space \( S \) is a quotient space of \( C \), and geodesics in \( S \) correspond to those geodesics in \( C \) that are of shortest lengths amongst all possible pairs in the two respective orbits. In other words, given two closed curves \( v_0, v_1 \in C \), we seek the shortest geodesic connecting their orbits \( C_{v_0} \) and \( C_{v_1} \). Mathematically, we seek a path in \( C \) whose length is given by \( d_S(v_0, v_1) \) where \( d_S(v_0, v_1) = \min_{u_0 \in C_{v_0}, u_1 \in C_{v_1}} d_C(u_0, u_1) \). Since the action of \( S^1 \) and \( SO(3) \) on \( C \) is isometric, we can restate this minimization as:

\[
\min_{\theta \in S^1, O \in SO(3)} d_C(\theta \circ Ov_0, v_1)
\]

We do not have an analytical solution for this minimization, so we are going to perform this numerically using an iterative technique. Let \( \theta(O) \in S^1 \times SO(3) \) be the current estimates of the minimizer, and let \( \psi(t, f, \tilde{v}_0) \) denote a geodesic flow starting from \( \tilde{v}_0 = \theta \circ Ov_0 \in C \), in the direction \( f \in T_{\tilde{v}_0}(C) \), and parameterized by \( t \in \mathbb{R}_+ \). The previous section discussed a procedure for finding \( f \) such that \( \psi(1, f, \tilde{v}_0) = v_1 \). We can write a four-dimensional subspace \( T_{\tilde{v}_0}(S^1 \times SO(3)) \), a subspace of \( T_{\tilde{v}_0}(C) \), such that flows in that direction do not change the shape of \( \tilde{v}_0 \). The projection of \( f \) into that subspace provides the best incremental rotation and shift (for re-parametrization) for solving the minimization in Eqn. 4.4. A repeated application of this idea - form a
four-dimensional subspace at the current $\tilde{v}_0$ and find the best incremental rotation-shift – leads to the optimal solution. This idea is pictorially illustrated in Figure 4.1.

Define,

$$E_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then, the functions $g_i(s) = E_i v_0(s), i = 1, 2, 3,$ and $g_4(s) = dv_0/da(s)$ span the desired four-dimensional subspace of $T_{v_0}(C)$. Forming an orthonormal basis of this subspace, using Gram-Schmidt procedure on $g_i$s, we get four basis elements $h_1, h_2, h_3$ and $h_4$. The projection of $f$ on this subspace is given by:

$$\sum_{i=1}^{4} \beta_i h_i,$$

where $\beta_i = \langle f, h_i \rangle$.

Define $A_{ij} = \langle g_i, h_j \rangle$, and setting $\gamma = A^{-1} \beta$, we obtain the projection in terms of the $g_i$s. That is, the projection is given by $\sum_{i=1}^{4} \gamma_i g_i$. Then, the optimal incremental rotation is given by $\exp(\delta \sum_{i=1}^{3} \beta_i E_i)$ and the optimal incremental shift is given by $\delta \gamma_4$. This procedure is summarized next.

**Subroutine 8 (Geodesic on $S$).**

initialize $\theta \in S^1$, $O \in SO(3)$, and $\beta = [1 \ 1 \ 1 \ 1]^T$.

while $\|\beta\| > \epsilon$

compute $\tilde{v}_0 = \theta \circ O v_0$.

find $f = \text{FindGeodesic}(\tilde{v}_0, v_1)$ using Algorithm 1.

form functions $g_1(s) = E_1 f(s), g_2(s) = E_2 f(s), g_3(s) = E_3 f(s),$ and $g_4(s) = f'(s)$.

use Gram-Schmidt on $g_i$s to obtain $h_i$s.

form $\beta_i = \langle f, h_i \rangle$, $A_{ij} = \langle g_i, h_j \rangle$, and $\gamma = A^{-1} \beta$.

update $O = O \exp(\delta \sum_{i=1}^{3} \beta_i E_i)$ and $\theta = \theta + \delta \beta_4$, for a small $\delta > 0$.

end while

Shown in Figure 4.2 is an illustration of this idea. The left panel shows the geodesic path between the two shapes at the start of Subroutine 8, and the middle panel shows the final (shortest) geodesic. The rightmost panel shows the evolution of geodesic length as iterations in Subroutine 8 proceeds.

The incremental update in Subroutine 8 is a local procedure, and depending upon the initialization of $(\theta, O)$, there is a possibility of this procedure getting stuck in a local minima of the geodesic distance function (Eqn. 4.4). In case of shapes with rotational symmetries, or re-parametrization symmetries, it is understandable that this greedy procedure gets stuck in a local minima. Our experiments indicate that
the search over $SO(3)$ in this procedure reaches the global minima consistently, even though the minimizer may not be unique. Shown in the top row Figure 4.3 are two examples of evolution of geodesic length, when minimizing over $O \in SO(3)$, while keeping $\theta \in S^1$ fixed. Different curves correspond to random initial values of $O$. In each case, starting from random initial orientations of $v_0$, the greedy procedure achieves the minimum geodesic length.

The case of re-parametrization is different. In search for optimal $\theta$, the procedure gets stuck in local minima, with geodesics much longer than the desired shortest geodesic. To minimize over $\theta \in S^1$, we can use an exhaustive grid search, as depicted by the bottom row of Figure 4.3. In other words, we choose and fix $\theta$ and minimize over $SO(3)$; repeating this for all values of $\theta$ and selecting the resulting shortest path gives us the desired geodesic.

5. Summary. We have presented a differential geometric approach to studying shapes of closed curves in $\mathbb{R}^3$. The main tool presented in this study is the construction
of geodesic paths between arbitrary two curves on an appropriate space of closed curves. This construction is based on path-straightening, i.e. we construct an initial path between those two curves, and iteratively straighten it using the gradient of the energy \( E \). The limit point of this procedure is a geodesic path. Finally, to compare shapes of two curves, we remove the shape-preserving transformations (rotation and re-parametrization) using an iterative local search. We have presented step-by-step procedures for computing these geodesics, and have illustrated them using examples.

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