NONPARAMETRIC DATA ANALYSIS ON MANIFOLDS WITH APPLICATIONS IN MEDICAL IMAGING

By

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To my loving son Christopher J. Osborne!!!
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Nonparametric Statistical Analysis on Manifolds in the past twenty years has been rapidly developed and applied to Medical Imaging problems. During this time period, statisticians began working more and more with nonlinear object data, regarding their observations as points on manifolds. The idea of a data analysis on abstract metric spaces goes way back to a visionary paper by Fréchet (1948), where he suggested to analyze object data on separable metric spaces rather than just on linear spaces in an effort of accommodate a large variety of elements (objects) that need to be analyzed statistically. Although standard statistical methods used for Euclidean data cannot be applied directly to such data, methodology has been developed for statistical analysis of data on manifolds. In this body of work, we focus two different medical imaging problems.

The first problem corresponds to analyzing the CT scan data. In this context, we perform nonparametric analysis on the 3D data retrieved from CT scans of healthy young adults, on the Size-and-Reflection Shape Space $SR\Sigma^k_{3,0}$ of k-ads in general position in 3D. This work is a part of larger project on planning reconstructive surgery in severe skull injuries which includes preprocessing and post-processing steps of CT images. The preprocessing step, consists of the extraction the boundary of the bone structure from the CT slices while the post-processing steps consists of 3D reconstruction of the virtual skull from these bone extractions and smoothing. Next, we present our results for the Schoenbergs sample mean Size-and-Reflection Shape of k-ads in general position in $\mathbb{R}^3$ for the human skull based on these virtual reconstructions. The bootstrap distribution of the Schoenberg sample means 3D Size-and-Reflection Shape for a selected group of anatomic landmarks and pseudo-landmarks, are computed for 500 bootstrap resamples of the original 20 skulls represented by the 3 by k configurations, when $k = 9$ and $k = 17$. Finally, we report a confidence region for the Schoenberg mean configuration.
The next problem corresponds to analyzing MR diffusion tensor imaging data. Here, we develop a two-sample procedure for testing the equality of the generalized Frobenius means of two independent populations on the space of symmetric positive matrices. These new methods, naturally lead to an analysis based on Cholesky decompositions of covariance matrices which helps to decrease computational time and does not increase dimensionality. The resulting nonparametric matrix valued statistics are used for testing if there is a difference on average between corresponding signals in Diffusion Tensor Images (DTI) in young children with dyslexia when compared to their clinically normal peers. The results presented here correspond to data that was previously used in the literature using parametric methods which also showed a significant difference.


CHAPTER 1

INTRODUCTION

Over the past twenty years, there has been a rapid development in Nonparametric Statistical Analysis on Manifolds and Shape Spaces, applied to various real life problems and in particular, Medical Imaging problems. Nonparametric statistics on size-and-reflection shape manifolds in 2D or 3D, is performed for purposes of diagnostics, discrimination, and/or identification. In order to begin to understand some of these relatively new areas of research, one must first understand what is a manifold.

Some of the disciplines in which data on manifolds originally arose in are cartography, astronomy, geology, meteorology, physics, and biology. As a result of the computer revolution, mathematicians and statisticians are now able to store huge digital vectors in data libraries and run nonlinear statistical algorithms to explore all possible matches among the contents provided by these vectors. Digital images possibly accounts for the largest types of data available today in modern scientific fields such as bioinformatics, medical imaging, computer vision, pattern recognition, astrophysics, forensics, etc.

Statisticians are working more and more with nonlinear object data, regarding their observations as points on manifolds. The idea of a data analysis on abstract metric spaces goes way back to a visionary paper by Fréchet (1948)[1], where he suggested to analyze object data on separable metric spaces rather than just on linear spaces in an effort of accommodate a large variety of elements (objects) that need to be analyzed statistically. Fréchet’s plan gained momentum much later, essentially with the IT revolution, when his direction could be followed through also at the computational level.

Although standard statistical methods used for Euclidean data cannot be ap-
plied directly to such data, methodology has been developed for statistical analysis of data on manifolds. For example, when describing probability distributions, it is often important to have some measure of central tendency, such as the mean or median. A natural measure of central tendency for a probability measure $Q$ on a metric space $M$ with the distance $\rho$ is the Fréchet mean (Fréchet (1948) [1], Ziezold (1974) [2]) which is the minimizer of $F(p) = \int \rho^2(p, x) Q(dx)$, if the minimizer is unique. In general, the set of all such minimizers is called the Fréchet mean set.

Bhattacharya and Patrangenaru (2003) [3] showed in their earlier work that there are various manners in which to define the distance $\rho$ used to calculate the Fréchet mean, resulting in different definitions of the Fréchet mean. If $(M, g)$ is a $m$-dimensional connected and complete Riemannian manifold, meaning that it is associated with a complete Riemannian metric $g$, then the Fréchet mean (set) with respect to the geodesic distance $d_g$ under $g$ is defined to be the intrinsic mean (set). The extrinsic mean $\mu_{E}(Q) = \mu_{j, E}(Q)$ of a probability measure $Q$ on a manifold $M$ with respect to an embedding $j : M \to \mathbb{R}^m$ is the Fréchet mean associated with the restriction to $j(M)$ of the Euclidean distance in $\mathbb{R}^m$. In [3], it was shown that the extrinsic mean of $Q$ exists if the ordinary mean of $j(Q)$ is a nonfocal point of $j(M)$, i.e., if there is a unique point $x_0$ on $j(M)$ having the smallest distance from the mean of $j(Q)$. In this case $\mu_{j, E}(Q) = j^{-1}(x_0)$.

Historically, the first examples of data analysis on manifolds are due to Watson (1983)[4], for directional data analysis (data analysis on spheres and real projective spaces), to D.G. Kendall (1984) [5], for direct similarity shape data analysis (data analysis on complex projective spaces) and to Chang(1988)[6] for tectonic plates data analysis (data analysis on groups of rotations). During a second phase, data analysis on more complicated sample spaces, including on certain Lie groups (Kim(2000)[7]), Stiefel manifolds (Hendricks and Landsman (1998)[8]), on projective shape manifolds (products of real projective spaces) (see Patrangenaru (2001)[9]), or on affine shape manifolds (Grassmannians) (see Patrangenaru and Mardia (2003)[10]), has been in the focus of modern statistical methodology. The common features of all these sample spaces is that they are all homogeneous spaces. Given a homogeneous space, it is statistician’s choice of selecting an appropriate homogeneous Riemannian structure on the sample space, that in her or his view would best address the data analysis in question. Although the areas of directional data analysis or shape data analysis that dominated object data analysis in
its initial phase, the homogeneous spaces were considered as sample spaces
and compact. In recent years, the attention has turned to brain imaging data and
size-and-shape data analysis in structural genomics for which the sample spaces
considered are of noncompact type (see Bandulasiri et al.(2009)[11], Bandulasiri et
al.(2009a)[12]). The ultimate objective of this paper is to give a concrete example of
nonparametric data analysis on a noncompact Riemannian homogeneous space of
nonconstant curvature.

Statistical analysis on general homogeneous spaces was first considered in the
context of density estimation with the ground breaking paper by R. J. Beran(1968)[13].
This line of research was also pursued for function estimation on Lie groups via
and Kim (2008)[16]. Such methodologies had in mind applications in medical
imaging, robotics, and polymer science (see Yarman and Yazici(2005)[17], Yarman
and Yazici(2003)[18], Koo and Kim (2008a)[19]). Projective shape data analysis
is also carried out on certain homogeneous spaces, products or real projective
spaces (see Patrangenaru et. al.(2010)[20]). Moreover, in the context of highest
interest of 3D human vision, this projective shape spaces have a Lie group struc-
tures, as shown in Crane and Patrangenaru (2011)[21], thus allowing to carry out
two sample tests for means on a Lie group. Data analysis taking values on ho-
mogeneous spaces that admit a structure of Riemannian homogeneous spaces of
noncompact type appeared in Diffusion Tensor Imaging (DTI) and in Cosmic Mi-
crowave Background (CMB) radiation (Schwartzman et. al. (2008)[22]). Both ar-
areas leads to analysis of random objects on the set of positive definite symmetric
matrices $\text{Sym}^+(p)$. From the perspective of data analysis on a homogeneous
space, the main statistical techniques used for DTI data analysis were parametric
in nature(Schwartzman(2006)[23], Schwartzman et. al.(2008)[22] and Schwartz-
man et. al.(2008a)[24]). Most recently, Haff et. al.(2012)[25], used the geometric ap-
proach of Lie group actions and Fourier analysis due to Helgason(1962)[26], which
is related to the approach in this paper.

The remaining portion of this body of work is organized as follows. In Chapter
2, we give an overview of what is a manifold and the properties of a manifold. In
addition, we introduce the general concepts of a lie group, homogeneous space,
homogeneous metric space, and Riemannian homogeneous space. In general, a
homogeneous space admits more Riemannian homogeneous structures (Tricerri
and Vanhecke (1983)[27]) and a metric classification theorem for connected, simply connected Riemannian homogeneous spaces were given in Patrangenaru(1994)[28] using the Cartan triple method. Chapter 3 deals with Nonparametric statistics on manifolds. Here, we provide a definition of the intrinsic mean $\mu_I(Q)$ and extrinsic mean $\mu_E(Q) = \mu_{j,E}(Q)$ of a probability measure $Q$ on a manifold $\mathcal{M}$ followed by nonparametric bootstrap confidence regions for the extrinsic mean. In Chapter 4, we introduce the size-and-reflection-shape space $SR^k_{p,0}$ of $k$-ads in general position in $\mathbb{R}^p$. In addition, the extrinsic analysis based on an Schoenberg equivariantly embedded in a space of symmetric matrices, allowing a nonparametric statistical analysis based on Schoenberg’s extrinsic means [11]. In Chapter 5, we recall the definition of a generalized Frobenius metric, the name of a homogeneous structure on $Sym^+(p)$ that was coined by Schwartzman (2006)[23]. The generalized Frobenius metric was first used in the DTI literature by Fletcher(2004)[29], and soon after by Moakher (2005)[30], by Arsigny et. al. (2006,2006a)[31, 32], by Deriche et. al.(2006)[33] and by Pennec et. al.(2006)[34] Furthermore, we give results concerning intrinsic means of random objects on Hadamard manifolds. In Chapter 6, we give a comprehensive application of size-and-reflection shape space $SR^k_{3,0}$ of $k$-ads in general position in 3D. For surgery planning, a more natural approach is to take into account the size in addition to shape when analyzing the CT scan data. In this context, one performs a nonparametric analysis on the 3D data retrieved from CT scans of young adults, on the on size-and-reflection shape space $SR^k_{3,0}$ of $k$-ads in general position in 3D. In this work, most of our focus includes preprocessing and post-processing steps of CT images as well as extrinsic analysis based on the Schoenberg equivariantly embedded [11]. Chapter 7 is concerned with Theorem 7.1, the main result of our following work in Osborne and Patrangenaru (2011) [35, 36]. Here, we prove the asymptotic normality of a two sample test statistic for intrinsic means on a Hadamard manifold that admits a simply transitive Lie group of isometries. As a consequence, in this section we also enounce Theorem 7.3, a nonparametric bootstrap result that can be applied in the two sample problem for intrinsic means on a Hadamard manifold, when the sample sizes are small. In Section 7.2 of Chapter 7, we inch closer to the data analysis in our paper, by detailing the two sample test for generalized Frobenius means on the space of symmetric matrices in terms of Cholesky decompositions, as shown in Theorem 7.5 and in Theorem 7.6. In Chapter 8, we apply our new
methodology to a concrete DTI example, using a small from a dataset previously analyzed by Schwartzman et al.(2008)[22], kindly provided by the main author of that paper. Our analysis consists of a voxelwise comparison of spatially registered DT images belonging to two groups of children, one with normal reading abilities and one with a diagnosis of dyslexia. We illustrate our methods in a single voxel that was found in Schwartzman et al.(2008)[22] to exhibit a strong difference between the two groups, using a parametric methodology. While Schwartzman et. al.(2008)[22] found that difference using parametric methods, the application herein shows that such differences may also be detected nonparametrically. Finally in Section 8.3 of Chapter 8, we give two new directions for future related research. It would be useful to extend the theoretical work in this paper to two sample tests on locally homogeneous Riemannian spaces, given that the intrinsic sample means are consistent, therefore for large samples, they would concentrate locally around the population intrinsic means. The other suggested direction is applied, for DTI based detection of branching points of brain vessels, etc.

1.1 Contributions

In this body of work, we focus two different medical imaging problems. The first problem corresponds to the following work Osborne et.al. (2011)[37] and Osborne et.al. (2012)[38] which includes preprocessing and post-processing steps of CT image analysis modelled on the Size-and-Reflection Shape Space. In this context, we performed a nonparametric analysis on the 3D data retrieved from CT scans of healthy young adults, on the on size-and-reflection shape space $SR^{3,k}_{3,0}$ of $k$-ads in general position in 3D. This work, as part of larger project on planning reconstructive surgery in severe skull injuries. The preprocessing step consists of the extraction the boundary of the bone structure from the CT slices whereas the post-processing step consists of 3D reconstruction of the virtual skull from these bone extractions. Numerous of methods and systems have been developed for object extraction (thresholding and segmentation) from 2D for 3D display [39, 40, 41, 42]. Visualization of 3D biomedical volume data (images) has traditionally been divided into two different techniques: surface rendering [43, 44, 45, 46, 47, 48] and volume rendering [49, 50, 51, 52, 53, 48, 54]. First, we explored thresholding the CT images and performing 3D reconstruction; however, the 3D reconstruction based
on thresholding the CT images did not yield very good visual 3D reconstructions (See Chapter 6). As a result, we turned are attention towards segmentation methods. In particular, we first find the bone structure for each CT slice by utilizing a fast total variation segmentation minimization algorithm to find the globally optimal solution (bone structure) for each CT slice Unger et. al. (2008) [55]. 3D reconstruction and smoothing was performed on segmented images in Matlab. Appendix C displays our results of the twenty 3D reconstructed virtual skulls after smoothing. One should note that our work here is based on surface rendering [43, 44, 45, 46, 47, 48]. Next we present our results for the Schoenberg sample mean size-and-reflection shape of k-ads in general position in $\mathbb{R}^3$ for the human skull based on these virtual reconstructions. In addition, the bootstrap distribution of the Schoenberg sample means 3D size-and-reflection shape for a selected group of anatomic landmarks and pseudo-landmarks, are computed for 500 bootstrap resamples of the original 20 skulls represented by the 3 by k configurations, when $k = 9, k = 17$. The 3 by k anatomic landmarks and pseudo-landmarks are displayed in Appendix A and Appendix B. Finally, we report a confidence region for the Schoenberg mean configuration.

The second problem corresponds to the following work Osborne and Patrangenaru (2011) [35, 36] which addresses much needed asymptotic and nonparametric bootstrap methodology for two-sample tests of data on Riemannian homogeneous manifolds with a simply transitive group of isometries. Here, we develop a two-sample procedure for testing the equality of the generalized Frobenius means of two independent populations on the space of symmetric positive matrices. These new methods, naturally lead to an analysis based on Cholesky decompositions of covariance matrices which helps to decrease computational time and does not increase dimensionality. The resulting nonparametric matrix valued statistics are used for testing if there is a difference on average between corresponding signals in Diffusion Tensor Images (DTI) in young children with dyslexia when compared to their clinically normal peers. The results presented here is for data that was previously used in the literature using parametric methods which also showed a significant difference.
CHAPTER 2

DATA ON MANIFOLDS

In recent years, the concept of manifolds has become central to many parts of non-parametric shape analysis because it allows more complicated structures to be expressed and understood in terms of the relatively well-understood properties of simpler spaces. In this section, we shall give a practical definition of a manifold which may seem to be quite abstract. Manifold is a notation obtained to generalize curves and surfaces in space to higher dimensions \([56]\). According to (Whitney’s embedding theorem) any smooth compact \(n\)-dimensional manifold, can be embedded as a close submanifold in \(\mathbb{R}^{2n+1}\). Based solely on Whitney’s embedding theorem, one might think it is enough to work only in the numerical space \(\mathbb{R}^n\). One should be caution of the potential danger of embedding a manifold in \(\mathbb{R}^n\), that is, even if a manifold is embeddable in \(\mathbb{R}^n\), the embedding is not always a natural one and does not always reveal the structure of the manifold. In other words, there is danger that it may conceal symmetry or other things that might have existed in the manifolds. For example, the sets of (1) all lines in the plane, (2) all planes in space, and (3) some particular patterns on a surface, turn out to be manifolds. If we stick only to \(\mathbb{R}^n\), then these manifolds may not come out.

2.1 What is a Manifold?

In differential geometry and topology, a manifold, \(\mathcal{M}\), is an mathematical space that is locally homeomorphic to Euclidean space (numerical space) \(\mathbb{R}^m\), but not necessarily global. Locally homeomorphic to Euclidean space means that every point has a neighborhood homeomorphic to an open Euclidean \(m\)-dimensional
ball,
\[ \mathcal{B}_r^m = \{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : x_1^2 + x_2^2 + \cdots + x_m^2 < r\}. \] (2.1)

In general, assume \((\mathcal{M}, \rho)\) is a metric space, with the metric topology \(\mathcal{T}_\rho\), generated by open balls in \(\mathcal{M}\)
\[ \mathcal{B}_r(x) = \{y \in \mathcal{M} : \rho(y, x) < r\}. \] (2.2)

The number of independent parameters, for which the transition map between two parametrizations are smooth. The dimension of the manifold, denoted by,
\[ \text{dim}(\mathcal{M}) = m, \] (2.3)
is given by the number of parameters needed in such a parametrization. A chart on \(\mathcal{M}\) is a homeomorphism of an open subset \(U \subseteq \mathcal{M}\) onto an open subset of \(\mathbb{R}^m\).

A \(m\)-dimensional chart \(x : U \rightarrow x(U) \subseteq \mathbb{R}^m\) is traditionally indicated by the pair \((U, x)\). If \(\mathcal{M}\) is connected metric space, then by the invariance of the domain theorem, any other chart \((V, y)\) is also \(m\)-dimensional. In most practical applications, a manifolds are generally taken to have a fixed dimension, and such a space is called an \(m\)-dimensional pure manifold; however, some authors does admit to manifolds where at different points the dimensionality of the manifold changes. For example, consider the sphere which has a constant dimension of two and is therefore a pure manifold whereas the disjoint union of a sphere and a line in three-dimensional space is not a pure manifold. Henceforth, we assume a manifold to be connected metric space having a fixed dimensionality at every point on \(\mathcal{M}\).

### 2.2 Properties of Manifolds

The structure of a manifold is often describe by a collection of coordinate charts which form an atlas, in analogy with an atlas consisting of charts of the surface of the Earth. The spherical Earth is navigated using flat maps or charts, collected in an atlas. Similarly, mathematical maps (coordinate charts) can be used to describe a differentiable manifold.
2.2.1 Charts

A coordinate map, a coordinate chart, or simply a chart, of a manifold is an invertible map between a subset of the manifold and a simple space such that both the map and its inverse preserve the desired structure [56].

2.2.2 Atlas

Describing most manifolds often requires more than one chart. A single chart is adequate for only the simplest manifolds. A collection of charts which covers a manifold is called an atlas. One should note that an atlas is not unique. All manifolds can be covered multiple ways using different combinations of charts.

**DEFINITION 2.1** A collection \( \mathcal{A} = \{(U_\alpha, \phi_\alpha)\} \) of \( \mathbb{R}^m \)-valued charts on \( \mathcal{M} \) is called an atlas of class \( C^r \) if the following conditions are satisfied:

(i) \( \cup_\alpha U_\alpha = \mathcal{M} \).

(ii) The sets of the form \( \phi_\alpha(U_\alpha \cap U_\beta) \) are all open in \( \mathbb{R}^m \).

(iii) Whenever \( U_\alpha \cap U_\beta \) is not empty, then the map \( \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta) \) is a \( C^r \) diffeomorphism.

Hence, a manifold with a \( C^r \) structure is called a \( C^r \) differentiable manifold or simply a \( C^r \) manifold.

2.2.3 Transition Maps

Charts in an atlas may overlap and a single point of a manifold may be represented in several charts. If two charts overlap, then parts of them represent the same region of the manifold, just as a map of Florida and a map of Georgia may both contain Tallahassee. Given two overlapping charts, a transition function, \( \phi \), can be defined which goes from an open ball in \( \mathbb{R}^m \) to the manifold and then back to another (or perhaps the same) open ball in \( \mathbb{R}^m \). The resulting map is called a change of coordinates, a coordinate transformation, a transition function, or a transition map.

Consider a the set of all differentiable curves \( c(t) \) on a \( C^r \) manifold:

A differentiable curve on a manifold \( \mathcal{M} \) is a differentiable function from an open
interval to $M$. Two curves $c_a(t), a = 1, 2$ defined on a neighborhood of $0 \in \mathbb{R}$ are tangent at $0$ if $c_1(0) = c_2(0) = p$ and there is a chart $(U, \phi)$ around $p$ such that $(\phi \circ c_1)'(0) = (\phi \circ c_2)'(0)$.

**Definition 2.2** The set of all curves tangent at $0$ to the curve $c$ is called tangent vector $v_p$ at $p = c(0)$, and is labeled $v_p = \frac{dc}{dt}(0)$.

If $M$ is a submanifold of $\mathbb{R}^n$, then two curves $c_1(t), c_2(t)$ on $M$ with $c_1(0) = c_2(0) = p$ have the same tangent vector in the sense of multivariable calculus, if $\frac{dc_1}{dt}(0) = \frac{dc_2}{dt}(0)$. Therefore, if $\phi : U \to \mathbb{R}^m$ is a chart around $p$, then $\frac{d(\phi \circ c_1)}{dt}(0) = \frac{d(\phi \circ c_2)}{dt}(0)$.

**Definition 2.3** The tangent space $T_pM$ at a point $p$ of a manifold $M$ is the set of all vectors tangent to curves in $M, c : (-\varepsilon, \varepsilon) \to M$ with $c(0) = p$.

**Definition 2.4** The tangent bundle space $TM$ of a manifold $M$ is the disjoint union of all tangent spaces, $T_pM$, at different points of $M$:

$$TM = \bigsqcup_{p \in M} T_p M. \tag{2.4}$$

We define the projection map

$$\Pi : TM \to M \tag{2.5}$$

by sending $v_p \in T_p M$ to $\Pi(v_p) = p$. Note that, we have $\Pi^{-1}(p) = T_p M$.

The tangent bundle space is often denoted by the triple $(TM, \Pi, M)$, where the projection map, $\Pi$, associates to each tangent vector to its base point, $\Pi(v_p) = p$.

The tangent bundle space, $TM$, has a natural structure of $2m$-dimensional $C^r$ manifold; in addition, the projection map $\Pi$ is of class $C^r$. A vector field $V$ is a differentiable section of $\Pi$, that is $V : M \to TM$, is a differentiable function for which $\Pi \circ V = Id_M$. An equivalent definition is that a vector field on $M$ is an assignment to every point of $M$ of a tangent vector to $M$ at that point, that is differentiable as a function of that point. That is, for each $p \in M$, we have a tangent vector $V(p) \in T_p M$.

An example of vector field on the sphere is suggested by the velocity of the wind at a given moment at different points on the surface of a planet.

**Definition 2.5** An embedding of a $C^r$ manifold $M$ in an Euclidean space $\mathbb{R}^m$ is a differentiable one-to-one function $j : M \to \mathbb{R}^m$, for which
(i) the differential $d_p j$ is a one-to-one function from $T_p \mathcal{M}$ to $\mathbb{R}^m$, and

(ii) $j$ is a homeomorphism from $\mathcal{M}$ to $j(\mathcal{M})$ with metric topology induced by the Euclidean distance.

The requirement of $j$ be differentiable simply mean that one can take the derivative of $j$.

Let $\text{Sym}(m+1, \mathbb{R})$ be the space of $(m+1) \times (m+1)$ symmetric matrices.

**EXAMPLE 2.6** The Veronese-Whitney map $j : \mathbb{R}P^m \rightarrow \text{Sym}(m+1, \mathbb{R})$, given by

$$j([x]) = xx^T, \quad x^T x = 1$$

(2.6)

is an embedding of the real projective space $(\mathbb{R}P^m)$ in an Euclidean space of symmetric matrices. The formal definition of the real projective space $\mathbb{R}P^m$ is given on page 18.

**DEFINITION 2.7** Assume the Lie group $G$ acts on both $\mathcal{M}$ and on $\mathbb{R}^m$ (refer to definition 2.8). A $G$-equivariant embedding $j$ of $\mathcal{M}$ in $\mathbb{R}^m$ is an embedding $j$ that is compatible with the two group actions, that is

$$j(g \cdot p) = g \cdot j(p).$$

(2.7)

A manifold $\mathcal{M}$ is said to be a homogeneous space if there is a Lie group $K$ acting transitively on $\mathcal{M}$. If $\mathcal{M}$ is such a homogeneous space, and the transitive group $K$ is known, a $K$-equivariant embedding $j$ of $\mathcal{M}$ is called simple equivariant embedding.

### 2.3 Lie Groups and Homogeneous Spaces

#### 2.3.1 Lie Groups

Lie groups are $C^r$ manifolds and, therefore, can be studied using differential calculus. One of the key ideas in the theory of Lie groups, from Sophus Lie, is to replace the global object, the group, with its local or linearized version, which Lie himself called its “infinitesimal group” and which has since become known as its Lie algebra.

**DEFINITION 2.8** A Lie group is a group which is also a differentiable manifold, $G$, with the property that the group operations of multiplication and inversion are smooth (differentiable) maps.
The group operations multiplication and inversion are compatible with the smooth algebraic structure of the group, for which the group multiplication:

\[ \alpha : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}, \text{ with } \alpha(x, y) = xy \]  

(2.8)

and the operation of taking the inverse:

\[ g \rightarrow g^{-1} \]  

(2.9)

are differentiable functions. These two requirements can be combined to the single requirement that the mapping

\[ (x, y) \mapsto x^{-1}y \]  

(2.10)

be a smooth mapping of the product manifold \( \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G} \).

**DEFINITION 2.9** \( \alpha \) is a left differentiable action on \( \mathcal{M} \) if

\[
\alpha \text{ is a differentiable function,}
\]

\[
k \cdot (h \cdot p) = (k \cdot h) \cdot p, \; \forall \; k, h \in \mathcal{G}, \; \forall \; p \in \mathcal{M},
\]

\[
1_K \cdot p = p, \; \forall \; p \in \mathcal{M}.
\]

(2.11)

### 2.3.2 Homogeneous Spaces

Most of the times, the sample spaces considered are metric spaces with a manifold structure, making them look locally, but not necessarily global like a numerical space \( \mathbb{R}^p \). Moreover, to enable comparisons of distributions, as a minimal requirement, a homogeneity property is sought for such manifolds; it is assumed that any two observations (points) on such a sample space \( (\mathcal{M}, \rho) \) can be brought into coincidence by an isometry. Formally this property amounts to the transitivity of the left action of the isometry group on the sample space \( (\mathcal{M}, \rho) \).

**DEFINITION 2.10** If \( (\mathcal{K}, \cdot) \) is a group with the identity element \( e \in \mathcal{K} \), and \( \mathcal{M} \) a set, then

- a map \( \alpha : \mathcal{K} \times \mathcal{M} \rightarrow \mathcal{M} \) is called a left action of \( \mathcal{K} \) on \( \mathcal{M} \) if the following two
properties are satisfied:
\[
\alpha_{k \cdot h} = \alpha_k \circ \alpha_h \quad \forall k, h \in K,
\]
and
\[
\alpha_e = \text{identity map } \mathcal{M} \to \mathcal{M},
\]
and for all \( k \in K \), we define \( \alpha_k : \mathcal{M} \to \mathcal{M} \), by \( \alpha_k(x) := \alpha(k, x) \).

- Given a left-action \( \alpha : K \times \mathcal{M} \to \mathcal{M} \), for \( x \in \mathcal{M} \), the isotropy group at \( x \), is the subgroup \( K_x = \{ k \in K : \alpha_k(x) = x \} \subset K \), and the orbit of \( x \) is the set \( K(x) = \{ \alpha_k(x), k \in K \} \).

- The left action \( \alpha \) is called transitive if for all \( x_1, x_2 \in \mathcal{M} \), there exists a \( k \in K \) such that \( \alpha_k(x_1) = x_2 \). If the action \( \alpha \) is transitive, then the manifold \( \mathcal{M} \) is called \( K \)-homogeneous space, or simply, homogeneous space.

Some may feel as if the algebraic definition 2.10 is too abstract for object data analysis, whereas here, we are interested in a analyzing distribution on metric spaces \((\mathcal{M}, \rho)\) only. For this reason, we will only consider \( K \)-homogeneous spaces, relative to a subgroup \( K \) of the group of isometries \( I(\mathcal{M}, \rho) \), where the group structure is given by the composition of functions. In such a situation, the action \( \alpha : K \times \mathcal{M} \to \mathcal{M} \) is \( \alpha(f, x) = f(x) \). Such a homogeneous space will be called homogeneous metric space. Euclidean spaces, spheres, projective spaces groups of rotations, and Stiefel manifolds endowed with extrinsic distances, that were historically considered first as sample spaces for object data analysis, are all homogeneous metric spaces.

The homogeneous space considered in this paper is the set \( Sym^+(p) \) of \( p \times p \) positive definite matrices. \( Sym^+(p) \) is the sample space for CMB data analysis, when \( p = 2 \), or for DTI data analysis, when \( p = 3 \).

**Proposition 2.11** If \( GL^+(p, \mathbb{R}) \) is the group of matrices of positive determinant and let \( \alpha : GL^+(p, \mathbb{R}) \times Sym^+(p) \to Sym^+(p) \) be given by
\[
\alpha(k, x) := \alpha_k(x) = k x k^T,
\]
then \( \alpha \) is a differentiable transitive left action of \( GL^+(p, \mathbb{R}) \) on \( Sym^+(p) \), and the isotropy group at \( I_p \) is the special orthogonal group \( SO(p) \).
Two general methods of endowing a distance \( \rho \) on a Riemannian metric \( g \) on \( M \) one may first endow \( M \) with a structure of \( K \)-homogeneous space, and, next find an isometric embedding of the Riemannian homogeneous manifold \( (M, g) \) into a \( K \)-homogeneous metric space for the left action \( \alpha : K \times M \to M \)

\[
\text{DEFINITION 2.12} \quad \text{An isometric embedding of the metric space } (M, \rho) \text{ is an injective map } J : M \to \mathbb{R}^D \text{ such that } \| J(x) - J(y) \| = \rho(x, y), \forall x, y \in M. \text{ If in addition } (M, \rho) \text{ is a } K\text{-homogeneous metric space for the left action } \alpha : K \times M \to M \text{ such an embedding is called isometric equivariant embedding.}
\]

\[
\text{DEFINITION 2.13} \quad \text{If } f : M \to M \text{ is differentiable, and } x \in M \text{ the differential of } f \text{ at the point } x, dxf : T_xM \to T_{f(x)}M, \text{ is defined by}
\]

\[
dxf \left( \frac{dc}{dt} \right) (0) = \frac{df \circ c}{dt} (0), \text{ where } c : (-\varepsilon, \varepsilon) \to M, \ c(0) = x. \quad (2.15)
\]

Given a Riemannian metric \( g \) on \( M \), an isometric embedding of the Riemannian manifold \( (M, g) \) is an embedding \( J : M \to \mathbb{R}^D \), such that

\[
g_x(\xi, \eta) = d_xJ(\xi)^Td_xJ(\eta), \forall x \in M, \forall \xi, \eta \in T_xM.
\]

A Riemannian metric \( g \) on \( M \) is called \( K\)-invariant if \( \alpha_k \) is an isometry for all \( k \in K \); i.e. if

\[
g_{\alpha_k}(x)((d_x\alpha_k)(\xi), (d_x\alpha_k)(\eta)) = g_x(\xi, \eta), \ \forall x \in M, \ \forall \xi, \eta \in T_xM. \quad (2.16)
\]

The existence of an isometric embedding of a Riemannian manifold in \( \mathbb{R}^D \) given by Nash (1956) [57] does not provide explicitly the equations of such an embedding. The existence of an isometric embedding of a Riemannian homogeneous...
space is also highly non-trivial (see Moore(1976)[58]). In absence of an equivariant embedding of a homogeneous $K$-space, one has to look for $K$-invariant Riemannian structures on $M$. The following is a basic result on Riemannian homogeneous spaces can be found in Kobayashi and Nomizu (1963, p.154)[59].

**REMARK 2.14** Isometric equivariant embeddings were previously used in statistics for two sample tests for means. The inclusion of the round sphere $S^{N-1}$ as a subset of $\mathbb{R}^N$ is an example of isometric equivariant embedding that was used for directional data two sample tests (see Beran and Fisher (1998)[60]). The map $J : \mathbb{R}P^{N-1} \rightarrow Sym(N), J([x]) = \frac{xx^T}{x^Tx}$, is an isometric equivariant embedding that was used for paired two sample tests for mean projective shapes by Crane and Patrangenaru (2011)[21]. The map $J : \mathbb{C}P^{k-2} \rightarrow S(k-1, \mathbb{C}), J([z]) = zz^*, z^*z = 1$ is an isometric equivariant embedding that was first considered by Kent(1992) [61] for direct similarity planar shape analysis. In these examples the sample spaces are compact. It is difficult to find an isometric equivariant embedding of a noncompact homogeneous space. For example, for the set $H = \{x = (x_1, \ldots, x_N, x_{N+1}) \in \mathbb{R}^{N+1}, -\sum_{j=1}^N x_j^2 + x_{N+1}^2 = 1\}$, with the distance $\rho(x, y) = -\sum_{j=1}^N (x_j - y_j)^2 + (x_{N+1} - y_{N+1})^2$, there no explicit isometric embedding into an Euclidean space. The sample space $Sym^+(p)$ is also noncompact, so we will consider another method to provide this space with a structure of homogeneous metric space.

**PROPOSITION 2.15** Let $\alpha$ be an action of a Lie group $K$ on a manifold $M$, with the isotropy group $H$ at the point $x_0 \in M$ and a scalar product $g_{x_0}$ on $T_{x_0}M$. Given another point $x \in M$, there is a $k \in K$, with $\alpha_k(x_0) = x$. Furthermore we define a scalar product on $T_xM$ by

$$g_x(\xi, \eta) = g_{x_0}(d_x\alpha_{k^{-1}}(\xi), (x_0, d_x\alpha_{k^{-1}}(\eta))). \quad (2.17)$$

Here, $g_x$ is well defined (does not depend on the choice of $k$), if and only if

$$g_{x_0}(d_{x_0}\alpha_h(\xi), d_{x_0}\alpha_h(\eta)) = g_{x_0}(\xi, \eta), \forall h \in H, \forall \xi, \eta \in T_{x_0}M, \quad (2.18)$$

and in this case, $x \rightarrow g_x$ is a $K$-invariant Riemannian metric on $M$, which uniquely extends $g_{x_0}$ to a $K$-invariant Riemannian metric on $M$.

**DEFINITION 2.16** The $K$-invariant Riemannian metric on $M$ in equation (2.17) will be called canonical metric associated with $g_{x_0}$.

Recall that if $U$ is an open set in $\mathbb{R}^D$, then the tangent space at a point $x \in U$ is $T_xU = \{x\} \times U$, and the the tangent space of $U$ is $TU = \bigcup_{x \in U} T_xU = U \times \mathbb{R}^D$. In
particular, if \( x \in Sym^+(p) \) and \( T_xSym^+(p) = \{x\} \times Sym(p) \) then tangent space \( TSym^+(p) \) is \( Sym^+(p) \times Sym(p) \).

Bearing this in mind, given a tangent vector \((x, u) \in TSym^+(p)\), and referring to the action in equation (2.14), we get

\[
(d_x\alpha_k)((x, u)) = (kxk^T, kuk^T), \forall (x, u) \in TSym^+(p).
\]

(2.19)

The Frobenius scalar product on \( Sym(p) \) is given by \( u \cdot_F v = Tr(uv) \).

**DEFINITION 2.17** The generalized Frobenius metric on \( Sym^+(p) \) is the canonical metric associated with the scalar product \( g_{Ip} \) on \( T_{Ip}Sym^+(p) \) given by

\[
g_{Ip}((I_p, u), (I_p, v)) =: u \cdot_F v = Tr(uv).
\]

(2.20)

**REMARK 2.18** According to Proposition 2.15, the generalized Frobenius metric is well defined, since the isotropy group of the action (2.14) is special orthogonal group \( SO(p) \) and the condition (2.18) is automatically satisfied.

Élie Cartan introduced following notion of simply transitive group (see Cartan (1945, pp.275-293)[62]).

**DEFINITION 2.19** An action \( \alpha \) of a Lie group \( G \) on a manifold \( M \) is simply transitive if for any two points \( x_1, x_2 \in M, x_1 \neq x_2 \), there is a a unique \( k \in G \), such that \( \alpha_k(x_1) = x_2 \).

**REMARK 2.20** Beran and Fisher (1998)[60] and Mardia and Patrangenaru (2005)[63] reduced a two sample problem for mean location on a certain homogeneous space (a sphere, or a projective space \( \mathbb{R}P^m \)) to a one sample problem on a Lie group acting on that homogeneous space \( (SO(3), \text{respectively} \ SO(m + 1)) \). This procedure is local, and has the drawback that it increases the dimensionality of the manifold where the data analysis is performed, since the dimension of a transitive Lie group \( K \) acting transitively on a manifold \( M \), is in at least higher than the dimension of \( M \). Therefore, wherever the data is on a \( K \)-homogeneous space, it is useful to check on the existence of a subgroup \( G \) of \( K \) that is simply transitive with the induced action \( \alpha|_{G \times M} \). In this case, the \( \text{dim} \ G = \text{dim} \ M \), and the procedure does not increase the dimensionality.

**LEMMA 2.21** A left action \( \alpha \) of a group \( G \) on \( M \) is transitive, if and only if there is a point \( x_0 \in M \), such that the orbit \( G(x_0) = M \).
For proof of Lemma 2.21, assume \( x_1, x_2 \in \mathcal{M} \), are arbitrary. From the hypothesis there exist \( t_1, t_2 \in \mathcal{G} \), such that \( x_1 = \alpha_{t_1}(x_0) \) and \( x_2 = \alpha_{t_2}(x_0) \). From the properties of the action \( \alpha \), we obtain
\[
x_0 = \alpha_{t_1^{-1}}(x_1) \quad \text{therefore} \quad \alpha_{t_2}(\alpha_{t_1^{-1}}(x_1)) = x_2,
\]
which can be also written as \( \alpha_{t_2t_1^{-1}}(x_1) = x_2 \); that is, there exists a \( t = t_2t_1^{-1} \in \mathcal{G} \), such that \( \alpha_t(x_1) = x_2 \). This proves the transitivity of the group action \( \alpha \) of \( \mathcal{G} \) on \( \mathcal{M} \).

**PROPOSITION 2.22** The group \( T^+(p, \mathbb{R}) \) of lower triangular \( p \times p \), matrices with positive diagonal entries, with the restriction of the action 2.14 to \( T^+(p, \mathbb{R}) \times \text{Sym}^+(p) \), is transitive on \( \text{Sym}^+(p) \).

The proof is immediate, since by the Cholesky decomposition of a positive definite covariance matrix \( \Sigma \), there exist a unique matrix \( t \in T^+(p, \mathbb{R}) \) such that \( \Sigma = tt^T \). We will set \( t = c(\Sigma) \). This is same as saying that, given the action \( \alpha \) in (2.14), for any matrix \( \Sigma \in \text{Sym}^+(p) \), \( \exists t \in T^+(p, \mathbb{R}) \) such that \( \alpha_t(I_p) = \Sigma \). This shows that the orbit \( T^+(p, \mathbb{R})(I_p) = \text{Sym}^+(p) \), and the result follows from Lemma 2.21. ■

### 2.4 Examples of Lie Groups and Manifolds

The group \( GL(m, \mathbb{R}) \) of invertible matrices, with the matrix multiplication is a Lie group. The orthogonal group \( O(m) \), and its connected component, the special orthogonal group \( SO(m) \) is of all \( m \times m \) orthogonal matrices \( A \) such that \( \text{Det}(A) = 1 \), are also Lie groups. Some other Lie groups are the circle group, diffeomorphism group, conformal group, etc. In mathematics, the circle group, is the multiplicative group of all complex numbers with absolute value 1, that is, the unit circle in the complex plane.

\[
\mathcal{T} = \{ z \in \mathbb{C} : |z| = 1 \}.
\]

(2.21)

Note that the circle group is also the group of \( 1 \times 1 \) unitary matrices, \( U(1) \); these act on the complex plane by rotation about the origin.

The conformal group is the group of transformations from a space to itself that preserve all angles within the space [64, 65]. Several important conformal groups are:

- The conformal orthogonal group. If \( V \) is a vector space with a quadratic form \( Q \), then the conformal orthogonal group \( CO(V, g) \) is the group of linear transformations \( T(V) \) such that for all \( x \in V \), there exists a scalar \( \lambda \) such that

\[
Q(Tx) = \lambda^2Q(x).
\]

(2.22)
Note that the conformal orthogonal group is equal to the orthogonal group times the group of dilations \[64, 65\].

- The conformal group of the sphere. The group of conformal transformations of the \(n\)-sphere is generated by the inversions in circles \[64, 65\].

**Real Projective Space, \(\mathbb{R}P^m\):**

Consider the action of the multiplicative group \(\mathbb{R}^* = \mathbb{R} \setminus \{0\}\) on \(\mathbb{R}^{m+1}\), given by scalar multiplication

\[\alpha(\lambda, z) = \lambda z\] (2.23)

the quotient space is the \(m\)-dimensional real projective space \(\mathbb{R}P^m\), set of all lines in \(\mathbb{R}^{m+1}\) going through \(0 \in \mathbb{R}^{m+1}\). In other words, this defines the projection

\[\pi : \mathbb{R}^{m+1} \setminus \{0\} \to \mathbb{R}P^m,\] (2.24)

which is obviously a surjection (or onto). For two points \(x = (x_1, \ldots, x_{n+1})\), \(y = (y_1, \ldots, y_{n+1})\) in \(\mathbb{R}P^m\), their images given by \(\pi\) are the same if and only if there exists a nonzero number \(a \in \mathbb{R}\) such that \(y_i = ax_i\), for \((i = 1, \ldots, m + 1)\) \[56\]. In this case, if we denote this by \(x \sim y\), we can see \(\sim\) gives an equivalence relation on \(\mathbb{R}P^m\).

If \(m = 2\), it simple to show that \(\mathbb{R}P^2\) is a manifold. Note that \(\mathbb{R}P^2\) is a metric space with the distance between two lines in \(\mathbb{R}^{2+1}\) given by the measure of their acute angle. If \(u \in \mathbb{R}^{2+1} \setminus \{0\}\), \(u = (u^1, u^2, u^3)\), we label its orbit \(\mathbb{R}^*(u)\) by \([u] = [u^1 : u^2 : u^3]\). Consider the affine open subsets \(A_1, A_2, A_3\) given by

\[
A_1 = \{[1 : x^2 : x^3], x^2, x^3 \in \mathbb{R}\}, \\
A_2 = \{[x^1 : 1 : x^3], x^1, x^3 \in \mathbb{R}\}, \\
A_3 = \{[x^1 : x^2 : 1], x^1, x^2 \in \mathbb{R}\},
\] (2.25)

and the charts \(\{(A_i, \phi_i), i = 1, 2, 3\}\), given by

\[
\phi_1 : A_1 \to \mathbb{R}^2, \phi_1([u]) = \left(\frac{u^2}{u^1}, \frac{u^3}{u^1}\right) = (x_1, x_2), \\
\phi_2 : A_2 \to \mathbb{R}^2, \phi_2([u]) = \left(\frac{u^1}{u^2}, \frac{u^3}{u^2}\right) = (y_1, y_2), \text{ and} \\
\phi_3 : A_3 \to \mathbb{R}^2, \phi_3([u]) = \left(\frac{u^1}{u^3}, \frac{u^2}{u^3}\right) = (z_1, z_2).
\] (2.26)
It is obvious that $A_1 \cup A_2 \cup A_3 = \mathbb{R}P^2$.

Note that if $A_1 \cap A_2 = \{ [u] \in \mathbb{R}P^2, u^1 \neq 0, u^2 \neq 0 \}$ and if $[u] \in A_1 \cap A_2$ and $\phi_1([u]) = (x_1, x_2)$, and $\phi_2([u]) = (y_1, y_2)$, then since $x_1 = \frac{u^2}{u^1}$, $x_2 = \frac{u^3}{u^1}$, $y_1 = \frac{u^1}{u^2}$, $y_2 = \frac{u^2}{u^3}$ it follows that $y_1 = \frac{1}{x_1}$, $y_2 = \frac{x_1}{x_2}$, showing that $\phi_2^{-1}\phi_1$ is differentiable.

If $A_1 \cap A_3 = \{ [u] \in \mathbb{R}P^2, u^1 \neq 0, u^3 \neq 0 \}$ and if $[u] \in A_1 \cap A_3$ and $\phi_1([u]) = (x_1, x_2)$, and $\phi_3([u]) = (z_1, z_2)$, then since $x_1 = \frac{u^2}{u^1}$, $x_2 = \frac{u^3}{u^1}$, $z_1 = \frac{u^1}{u^3}$, $z_2 = \frac{u^2}{u^3}$ it follows that $z_1 = \frac{1}{x_2}$, $z_2 = \frac{x_2}{x_1}$, showing that $\phi_3^{-1}\phi_1$ is differentiable.

Finally, if $A_2 \cap A_3 = \{ [u] \in \mathbb{R}P^2, u^2 \neq 0, u^3 \neq 0 \}$ and if $[u] \in A_2 \cap A_3$ and $\phi_2([u]) = (y_1, y_2)$, and $\phi_3([u]) = (z_1, z_2)$, then since $y_1 = \frac{u^1}{u^2}$, $y_2 = \frac{u^2}{u^2}$, $z_1 = \frac{u^1}{u^3}$, $z_2 = \frac{u^2}{u^3}$ it follows that $z_1 = \frac{u_1}{y_2}$, $z_2 = \frac{1}{y_2}$, showing that $\phi_3^{-1}\phi_2$ is differentiable.

Thus, $\mathbb{R}P^2$ is an analytic manifold and similarly, one can show that $\forall m \in \mathbb{N}^*$, $\mathbb{R}P^m$ is an $m$-dimensional analytic manifold.

Similarly, if we consider the action of the multiplicative group $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ on $\mathbb{C}^{m+1}$, given by scalar multiplication

$$\alpha(\lambda, z) = \lambda z$$

(2.27)

the quotient space is the $m$-dimensional complex projective space $\mathbb{C}P^m$, set of all complex lines in $\mathbb{C}^{m+1}$ going through $0 \in \mathbb{C}^{m+1}$. One can show that $\mathbb{C}P^m$ is a $2m$ dimensional analytic manifold, using similar transition maps to those in the case of $\mathbb{R}P^m$.

### 2.5 Data on Manifolds

Manifolds are sets of points, curves, surfaces or higher dimensional curved objects. Data analysis on manifolds is an extension of multivariate data analysis of directional data analysis, of shape data analysis, etc. Due to the rapid development in Nonparametric Statistical Analysis on Shape Manifolds and Similarity Shape Spaces applied to various real life problems, there has been many advances in various disciplines. In particular, nonparametric statistics on size-and-similarity manifolds in 2D or 3D, is performed for purposes of diagnostics, discrimination, and/or identification on various Medical Imaging problems. In this section, we present some classical and current examples of data on manifolds.
2.5.1 CT Scan Data

A computed tomography (CT) scan uses X-rays to make detailed pictures of structures inside of the body. A CT scan is used to study all parts of the human body, such as the chest, belly, pelvis, or an arm or leg. CT scan can also take pictures of the body organs, such as the bladder, liver, lungs, pancreas, intestines, kidneys, and heart. In addition, CT scan can study spinal cord, blood vessels, and the bones. Figure 2.1 displays one CT scan of a head from a sample of 28 different patients.

2.5.2 DTI Data

Diffusion tensor imaging (DTI) is a magnetic resonance imaging (MRI) based technique that allows the measurement of the restricted diffusion of water in tissue in order to produce neural tract (fibers in the brain) images instead of using this data solely for the purpose of assigning contrast or colors to pixels in a cross sectional image. DTI stores information about the diffusion of the water molecules in the brain. Thus, DTI can trace subtle changes in the white matter associated with unusual brain wiring like in dyslexia or in schizophrenia, or brain diseases including multiple sclerosis or epilepsy. Diffusion MRI relies on the mathematics and physical interpretations of the geometric quantities known as tensors. The displacement of a water molecule at a certain location in the brain has probability distribution in space, and its diffusion \( D \) is half the covariance matrix of that distribution, which is a symmetric positive semidefinite matrix.

The diffusion matrix \( D = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix} \) at any given voxel in stored in the form of a column \((d_{11} \, d_{22} \, d_{33} \, d_{12} \, d_{13} \, d_{23})^T\), and as a result in the imaging of white matter, the location, orientation, and anisotropy of the fiber tracts can be measured. In Figure 8.1 we display DTI slices including a given voxel recorded in a control subject and a dyslexia subject.
Figure 2.1: One CT Scan of the head
Figure 2.2: DTI slice images of a control subject (left) and of a dyslexia subject (right)
CHAPTER 3

NONPARAMETRIC STATISTICS ON MANIFOLDS

Nonparametric statistics on manifolds has rapidly expanded in recent years due to applications in the bioinformatics, medical imaging, computer vision, pattern recognition, astrophysics, medicine, and in image analysis. Statistical inference for distributions on manifolds is now a broad discipline with wide ranging applications. There is a substantial amount literature in this field given by the following books Watson(1983) [4], Small(1996) [66], Kendall et al (1999) [67], Dryden and Mardia (1998) [68], Mardia and Jupp (2000) [69]. Most of this literature mention above focuses on parametric or semiparametric model. Here, we consider a general framework for nonparametric inference for location, as detailed by Bhattacharya and Patrangenaru (2002, 2003, 2005) [70, 3, 71] where general properties of extrinsic mean sets on general manifolds. In addition, we consider the asymptotic distributions of extrinsic sample means and confidence regions based on them. We provide classical CLT-based confidence regions as well as those based on Efron’s bootstrap (Efron(1982)[72]).

The intrinsic mean $\mu_I(Q)$ is the Fréchet mean of a probability measure $Q$ on a complete $m$-dimensional Riemannian manifold $\mathcal{M}$ endowed with the geodesic distance $d_g$ determined by the Riemannian structure $g$ on $\mathcal{M}$ (The Fréchet mean is not defined here. Refer to page 24). It is known that if a probability distribution $Q$ on a Riemannian manifold $(\mathcal{M}, g)$ is sufficiently concentrated, then the intrinsic mean $\mu_I(Q)$ exists. The intrinsic mean is easier to compute if the Riemannian manifold has zero curvature in a neighborhood containing $\text{supp}Q$ (Patrangenaru (2001)). In particular this is the case of distributions on linear projective shape spaces (Mardia and Patrangenaru (2002)). If the manifold has nonzero curvature around $\text{supp}Q$,
with respect to a metric of our choice, the intrinsic sample mean associated, can be approximated using fast algorithms such as the ones given in Groisser (2004) [73] or a Newton-Raphson type of algorithm such as the one given (see Pennec (1999) [74]). In practice, approximating the sample intrinsic mean could be a computationally intensive task for a poor chose of an algorithm. Due to the advances in computing power, this is becoming less of a major issue. In addition, it should be noted that there are no known conditions for the existence of the intrinsic mean of a distribution that is spread over even a slightly large region on a manifold [73].

However, the extrinsic mean $\mu_E(Q) = \mu_{j,E}(Q)$ of a probability measure $Q$ on a manifold $M$ w.r.t. embedded via $j : M \to \mathbb{R}^k$ into an Euclidean space, always exists and is unique if the ordinary mean of $j(Q)$ is a nonfocal point of $j(M)$, i.e., if there is a unique point $x_0$ on $j(M)$ having the smallest distance from the mean of $j(Q)$. Hence we focus our attention on extrinsic analysis.

In [70, 3], Bhattacharya and Patrangenaru (2002 and 2003) studied measures of location and dispersion for distributions on a manifold as Fréchet parameters associated with two types of distances on $M$. If $j : M \to \mathbb{R}^k$ is an embedding, the Euclidean distance restricted to $j(M)$ yields the extrinsic mean set and the extrinsic total variance.

The Fréchet mean of a probability measure $Q$ on a complete metric space $(M, \rho)$ is the minimizer of the function $F(x) = \int \rho^2(x, y)Q(dy)$, when such a minimizer exists and is unique (Fréchet (1948)). The extrinsic mean $\mu_E(Q) = \mu_{j,E}(Q)$ of a probability measure $Q$ on a manifold $M$ with respect to an embedding $j : M \to \mathbb{R}^k$ is the Fréchet mean associated with the restriction to $j(M)$ of the Euclidean distance in $\mathbb{R}^k$. In [3], it was shown that the extrinsic mean of $Q$ exists if the ordinary mean of $j(Q)$ is a nonfocal point of $j(M)$, i.e., if there is a unique point $x_0$ on $j(M)$ having the smallest distance from the mean of $j(Q)$. In this case $\mu_{j,E}(Q) = j^{-1}(x_0)$.

The extrinsic sample mean, for a given embedding $j : M \to \mathbb{R}^k$, straightforward to compute. If $Q$ is highly concentrated as in medical imaging examples in Bhattacharya and Patrangenaru (2003), the intrinsic and extrinsic means are virtually indistinguishable, should the arc and chord distances be close to each other around $\text{supp}Q$. 

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3.1 Analysis of Extrinsic Means on Manifolds

The set of probability measures on a manifold embedded in an Euclidean space that have an extrinsic mean is generic (open and dense) in the space of all probability measures on that manifold. If \( j : \mathcal{M} \to \mathbb{R}^N \) is an embedding of a manifold and \( X \) is a \( j \)-nonfocal random object on \( \mathcal{M} \), then \( \mu_j = j^{-1}(P_j(E(j(X)))) \).

The extrinsic covariance matrix \( \Sigma_j \) of \( X \) is the restriction to the tangent space at the extrinsic mean at the range of \( j, T_{\mu_j}j(\mathcal{M}) \) of

\[
\Sigma_j = d_{\mu_j}P_j\Sigma(d_{\mu_j}P_j)^T, \quad \text{where} \quad \Sigma = \text{Cov}(j(X)).
\]

Assume that \( X = (X_1, ..., X_n) \) are i.i.d. \( M \)-valued random variables whose common distribution is a nonfocal measure \( Q \) on \( (\mathcal{M}, j) \). If the mean \( \bar{j}(X) \) of the sample \( j(X) = (j(X_1), ..., j(X_n)) \) is a nonfocal point, the extrinsic sample mean is

\[
\bar{X}_E := j^{-1}\left(P_M(\bar{j}(X))\right) \equiv \mu_{E}(\hat{Q}_n),
\]

where \( \hat{Q}_n = n^{-1}\sum_{i=1}^{n} \delta_{X_i} \) is the empirical distribution.

A consistent estimator of \( \Sigma_j \) is the extrinsic sample covariance matrix

\[
G(j, X) = \left[ \sum \frac{d_{j(X)}P_j(e_b) \cdot e_a(P_j(\bar{j}(X)))}{a=1,...,m} \right] \cdot S_{j,n}
\]

\[
= \left[ \sum \frac{d_{j(X)}P_j(e_b) \cdot e_a(P_j(\bar{j}(X)))}{a=1,...,m} \right]^t,
\]

\( S_{j,n} = n^{-1}\sum (j(X_r)-\bar{j}(X))(j(X_r)-\bar{j}(X))^t \) is the sample covariance, and \((e_a(y), a = 1, \ldots, N)\) is an adapted orthoframe field around \( P_j(\bar{j}(X)) \).

3.1.1 Asymptotic distributions of extrinsic sample means

The direct sum \( \mathbb{R}^N = T_{P_{j(\mu)}j(\mathcal{M})} \oplus T_{P_{j(\mu)}j(\mathcal{M})}^\perp \) yields a decomposition of any vector \( u \in \mathbb{R}^N, u = u_{\text{tan}} + u_{\perp} \). The standardized X-mean is

\[
\bar{Z}_{j,n} := n^{1/2} \Sigma_j^{-1/2} \bar{X}_{\text{tan}},
\]

If \( \{X_r\}_{r=1,...,n} \) are i.i.d. random objects from a \( j \)-nonfocal distribution \( Q \) on \( \mathcal{M} \) and \( \Sigma_j \) of \( Q \) is finite. Then (a) the extrinsic sample mean \( \bar{X}_j \) has asymptotically a
\( \mathcal{N}_m(0, n^{-1}\Sigma_j) \) in \( T_{\mu_j, \mathcal{M}} \), and (b) if \( \Sigma_j \) is nonsingular, the \( j \)-standardized mean vector \( Z_{j,n} \) in (3.4) converges weakly to \( \mathcal{N}_m(0, I_m) \).

However, this result cannot be used in practice to construct confidence intervals since the population extrinsic covariance is unknown. For such purposes, instead of using \( \Sigma_j \), the consistent estimator \( G(j, X) \) is used. It then follows that

\[
n\|G(j, X)^{-\frac{1}{2}}\tan P_j(j(X)) (P_j(j(X)) - P_j(\mu))\|^2
\]

converges weakly to \( \chi^2_m \).

### 3.1.2 Confidence Regions for Extrinsic Means

Utilizing the above results, a large sample confidence region for \( \mu_j \) of asymptotic level \( 1 - \alpha \) is given by (a) \( C_{n,\alpha} := j^{-1}(U_{n,\alpha}) \), where \( U_{n,\alpha} = \{ \mu \in j(M) : n\|G(j, X)^{-\frac{1}{2}}\tan P_j(j(X)) - P_j(\mu)\|^2 \leq \chi^2_{m,1-\alpha} \} \); or by

(b) \( D_{n,\alpha} := j^{-1}(V_{n,\alpha}) \), where \( V_{n,\alpha} = \{ \mu \in j(M) : n\|G(j, X)^{-\frac{1}{2}}\tan P_j(j(X)) - P_j(\mu)\|^2 \leq \chi^2_{m,1-\alpha} \} \).

However, for many circumstances a large sample is not available. In those cases, an alternative method to obtain a confidence region for \( \mu_j \) is needed. One such approach is the use of nonparametric bootstrap. A 100(1 - \( \alpha \))% nonparametric bootstrap confidence region for \( \mu_j \) is \( D_{n,\alpha}^* := j^{-1}(V_{n,\alpha}^*) \) with \( V_{n,\alpha}^* \) given by

\[
V_{n,\alpha}^* = \{ \mu \in j(M) : n\|G(j, X^*)^{-\frac{1}{2}}\tan P_j(j(X^*)) - P_j(\mu)\|^2 \leq d_{1-\alpha}^* \} \tag{3.6}
\]

where \( d_{1-\alpha}^* \) is the upper 100(1 - \( \alpha \))% point of the values

\[
n\|G(j, X^*)^{-\frac{1}{2}}\tan P_j(j(X^*)) (P_j(j(X^*)) - P_j(\mu))\|^2
\]

among the bootstrap resamples.

This region has coverage error \( O_p(n^{-2}), \) so \( n_{\text{bootstrap}} \sim \sqrt{n_{\text{asymptotic}}} \)
3.2 Remarks on Nonparametric Statistics on Manifolds

In our new technological computing era, non-Euclidean data may possibly account for the largest type of data available today in modern scientific fields such as bioinformatics, medical imaging, computer vision, pattern recognition, forensics, etc. Patrangenaru (1998) [75], introduced extrinsic and intrinsic means on manifolds as location parameters for non-Euclidean data, following some definitions in the habilitation of Riemann and Gauss Egregium theorem as mentioned earlier in this chapter. As a result of large sample and nonparametric bootstrap analysis by Bhattacharya and Patrangenaru [3], [71], at the turn of the century, a flurry of papers in computer vision, statistical learning, pattern recognition, medical imaging and other computational intensive applied areas using these concepts followed. John Nash’s isometric embedding theorem [57] paved the way for other researchers to follow. J. D. Moore (1976)[58] developed an equivariant version of Nash’s isometric embedding theorem [57] which shows that any Riemannian manifold \((M, g)\), can be embedded in an Euclidean space isometrically, in such a way, that the group of isometries of \((M, g)\) is made of Euclidean distance preserving transformations, restricted to the image of such an embedding. One of the many future problems that needs to be solved stems directly from J. D. Moore (1976)[58] work, “How to find an equivariant embedding for any Riemannian manifold?”

In practical applications, it is known that it is easier to compute the extrinsic mean in the particular case of distributions on Grassmann manifolds (real or complex). It may be also be pointed out that if \(Q\) is highly concentrated, as in medical imaging data examples (Bhattacharya and Patrangenaru (2003)[3]) the intrinsic and extrinsic means are virtually indistinguishable. Given that the extrinsic analysis is computationally faster, and that there are as many Riemannian structures on an abstract manifold as there are embeddings, the current trend of using intrinsic analysis in computationally intensive applications, may shift towards an extrinsic approach.
CHAPTER 4

SIZE-AND-REFLECTION SHAPE SPACE

For all $p > 2, k > p$, a size-and-reflection-shape space $SR_{p,0}^k$ of $k$-ads in general position in $\mathbb{R}^p$, invariant under translation, rotation and reflection, is shown to be a smooth manifold and is equivariantly embedded in a space of symmetric matrices, allowing a nonparametric statistical analysis based on extrinsic means [11]. Nonparametric statistical analysis of landmark based size-and-shape data in which each observation $x = (x^1, \ldots, x^k)$ consists of $k > p$ points in $p$ dimension, called a $k$-ad, represents $k$ locations on an object. Landmark selections is generally made with expert help in the particular field of application. The objects of study can be anything for which two $k$-ads are deemed equivalent modulo a group $G$ of transformations depending on the features one wishes to compare and the method of collecting and recording of data. For example, one may consider the problem of discriminating between distributions of images of a normal and a diseased human organ. Here one may or may not discard the effects of magnification and differences that may arise because of variation in size or in equipment used or due to the manner in which images are taken and digitally recorded. An appropriate $G$ in this case would be the group of direct isometries, if size is taken into account, or the one generated by direct isometries and scaling, as proposed in a pioneering paper by Kendall (1984)[5] for measuring shapes, if the size is ignored. As an example, one may consider only $k$-ads in which the $k$ points are not all equal, and removes translation by centering the $k$-ad $x = (x^1, \ldots, x^k)$ to

$$\xi = (\xi^1, \ldots, \xi^k)$$

$$\xi^j = x^j - \overline{x}, \forall j = 1, \ldots, k.$$
Note that the set of all centered \( k \)-ads lie in a vector subspace \( L^p_k \) in \((\mathbb{R}^p)^k\) of dimension \( pk - p \), \( L^p_k = \{ \xi = (\xi^1, \ldots, \xi^k) \in (\mathbb{R}^p)^k : \xi^1 + \cdots + \xi^k = 0 \} \), and, \( L^p_k^* = L^p_k \setminus \{0\} \). The size-and-shape \([x]_S\) is the equivalence class, or orbit, of \( \xi = (\xi^1, \ldots, \xi^k) \) under rotations (Dryden and Mardia (1998, pp. 57)[68]). If the size is not relevant, its effect is removed by scaling \( \xi \) to unit size as

\[
u = \frac{\xi}{|\xi|}.
\]

The transformed quantity \( \nu \) is called a \textit{preshape}, and the set \( S(L^p_k^*) \) of all preshapes comprises a manifold of dimension \( pk - p - 1 \), namely, the unit sphere in \( L^p_k \) called the \textit{preshape sphere}.

Unfortunately, for \( p > 2 \) (and in particular \( p = 3 \)), both \( S\Sigma^k_p = L^p_k^*/SO(p) \) and \( \Sigma^k_p = S(L^p_k^*)/SO(p) \) have singularities, due to the fact that the actions of \( SO(p) \) on \( \mathbb{R}^{pk-p}\setminus\{0\} \), and on \( S^{pk-p-1} \) are not \textit{free}, i.e., there are \( k \)-ads \( x \) for which there exist different \( A, B \) in \( SO(p) \) such that \( Au = Bu \) (in the representation (4.2)), so that the orbits of \( \xi \), respectively of \( u \), and \( SO(p) \) are in general not one-to-one. For \( p = 3 \), this is the case if the \( k \) landmarks of a \( k \)-ad are collinear so that the \( k \) points in the preshape \( u \) lie on a straight line; note that \( \xi \), respectively \( u \), are left invariant by the nontrivial subgroup of rotations around this line as axis. The orbits of \( S\Sigma^k_p \), respectively of \( \Sigma^k_p \), are of different dimensions in different regions, thus failing to be differentiable manifolds in the usual sense. Thus, in dimension \( p \geq 3 \) the Kendall’s shape space \( \Sigma^k_p \) has no equivariant embedding of it that is known. The existence of singularities has been a well recognized problem since the inception of Kendall’s theory (Kendall et al. (1999)[67], Small (1996)[66]). This encumbers the statistical analysis of 3D images, if one adheres to Kendall’s shape spaces.

### 4.1 \( RS\Sigma^k_{p,0} \) Manifolds and \( SR\Sigma^k_{p,0} \) Manifolds in Higher Dimensions

The size-and-reflection-shape \([x]_{RS}\) of the \( k \)-ad \( x \) and the reflection shape \([x]_R\) of this \( k \)-ad, are, respectively, the \( O(p) \)-orbit of the centered configuration \( \xi \), and of the preshape \( u \) under the action \( A\xi = (A\xi^1, \ldots, A\xi^k) \) of the orthogonal group \( O(p) \).
(of $p \times p$ matrices $A$ satisfying $A^T A = I_p$) on the set of all centered $k$-ads (Dryden and Mardia (1998, p. 57))[68]. Thus

$$[x]_{RS} = \{ A \xi : A \in O(p) \}, \quad [x]_R = \{ A u : A \in O(p) \}. \quad (4.3)$$

The set of all size-and-reflection-shapes (respectively, reflection shapes) of $k$-ads in general position $\xi$, i.e. $k$-ads for which $\{ \xi_1, \ldots, \xi_k \}$ spans $\mathbb{R}^p$, is the size-and-reflection-shape space $SR\Sigma^k_{p,0}$ (respectively, the reflection shape space $R\Sigma^k_{p,0}$). Both of these spaces are manifolds since the action of an orthogonal matrix on $\mathbb{R}^p$ is uniquely determined by its action on a basis of $\mathbb{R}^p$, and a centered $k$-ad in general position includes such a basis. Therefore it is useful and easier to study shapes of configurations with respect to the full group of similarities. This group, $Sim(p)$, is the set of transformations of the Euclidean space $\mathbb{R}^p$, of the form

$$x' = B x + b, \quad B^T B = c I_p, \quad c > 0. \quad (4.4)$$

The reflection shape $[x]_R$ of a $k$-ad $x$, which is regarded here as a $p$ by $k$ matrix, is defined by $(4.3)$. The reflection shape manifold, introduced earlier, then is the set

$$R\Sigma^k_{p,0} = \{ [x]_R, \text{ x in general position } \} = \{ [x]_R, \text{rank}(x) = p \}. \quad (4.5)$$

The manifold dimension, codimension of the $O(p)$-orbits in $(\mathbb{R}^p)^k$, is $\dim R\Sigma^k_{p,0} = kp - \frac{k(p+1)}{2} - 1$.

The size-and-reflection-shape $[x]_{RS}$ of a $k$-ad $x$ in $\mathbb{R}^p$ is defined in $(4.3)$, and the size-and-reflection-shape space $SR\Sigma^k_{p,0}$ is the set

$$SR\Sigma^k_{p,0} = \{ [x]_{RS}, \text{ x in general position } \} = \{ [x]_{RS}, \text{rank}(x) = p \}. \quad (4.6)$$

Recall that, by the fundamental theorem of Euclidean geometry, any isometry of the Euclidean space is a linear transformation of $\mathbb{R}^p$ of the form

$$x' = B x + b, \quad B^T B = I_p, \quad (4.7)$$

therefore, if one centers the $k$-ad $x$ to $\xi = (x^1 - \overline{x}, \ldots, x^k - \overline{x}) \in I^p_k$, then $[x]_{RS} = [\xi]_{RS}$, and if a $k$-ad $x'$, having the same reflection-shape as $x$, differs from $x$ by the
isometry (4.7), then
\[ x' = Bx + b. \] (4.8)

Therefore \( \xi' = B\xi \), where \( \xi' \in L^k_p \) is obtained by centering the \( k \)-ad \( x' \). Thus, if we set \( L_{k,p,0} = \{ \xi \in L^k_p, \text{rk}\xi = p \} \), the manifold \( SR\Sigma^k_{p,0} \) can be represented as a quotient \( L_{k,p,0}/O(p) \) and the manifold dimension of \( SR\Sigma^k_{p,0} \) is \( kp - \frac{p(p+1)}{2} \).

## 4.2 Equivariant Embeddings of \( R\Sigma^k_{p,0} \) and \( SR\Sigma^k_{p,0} \)

In higher dimensions, using an approach based on well known result in multi-dimensional scaling (MDS) (see Schoenberg (1935) [76], Gower (1966) [77], Mardia et al. (1979, p. 397) [78], Bandulasiri and Patrangenaru(2005) [79] introduced the Schoenberg embedding of reflection shapes in higher dimensions. Let \( D = (d_{rs})_{r,s=1}^{k} \) be a distance matrix between \( k \) points and consider \( A \) and \( B \) defined by
\[ A = (a_{rs}), \quad a_{rs} = -\frac{1}{2} d^2_{rs}, \quad \forall r, s = 1, \ldots, k, \]
\[ B = \tilde{H}A\tilde{H}^T, \]
where \( \tilde{H} = I_k - k^{-1}1_k1_k^T \) is the centering matrix in \( \mathbb{R}^k \). A version of Schoenberg’s theorem suitable for our purposes is given below. Theorem 4.1 is an edited version of Theorem 14.2.1 in Mardia et al. (1979, pp. 397-398)[78].

**THEOREM 4.1**  
(a) If \( D \) is a matrix of Euclidean distances \( d_{rs} = \|x^r - x^s\|, x^j \in \mathbb{R}^p \), and we set \( X = (x^1 \ldots x^k)^T, k > p \), then \( B \) is given by
\[ b_{rs} = (x^r - \overline{x})^T(x^s - \overline{x}), r, s = 1, \ldots, k \] (4.9)

In matrix form, \( B = (\tilde{H}X)(\tilde{H}X)^T \).

(b) Conversely, assume \( B \) is a \( k \times k \) symmetric positive semidefinite matrix \( B \) of rank \( p \), having zero row sums. Then \( B \) is a centered inner product matrix for a configuration \( X \) constructed as follows. Let \( \lambda_1 \geq \cdots \geq \lambda_p \) denote the positive eigenvalues of \( B \) with corresponding eigenvectors \( x^{(1)}, \ldots, x^{(p)} \) normalized by
\[ x^{(i)}_T x^{(i)} = \lambda_i. \] (4.10)

Then the \( k \) points \( P_r \in \mathbb{R}^p \) with coordinates \( x^r = (x_{r1}, \ldots, x_{rp})^T, r = 1, \ldots, k \), where \( x^r \) is the \( r \)-th row of the \( k \times p \) matrix \( (x^{(1)} \ldots x^{(p)}) \), have center \( \overline{x} = 0 \), and \( B = \tilde{H}A\tilde{H}^T \) where \( A = -\frac{1}{2}(\|x^r - x^s\|)_r,s=1,\ldots,k \).
4.2.1 Schoenberg Embedding of the $R\Sigma^k_{p,0}$

Let $S(k, \mathbb{R})$ denote the set of all $k \times k$ real symmetric matrices. The Schoenberg embedding $J : R\Sigma^k_{p,0} \rightarrow S(k, \mathbb{R})$ is given by:

$$A = J([x]_R) = u^T u,$$

(4.11)

where $u = \frac{\xi}{\|\xi\|}$ and $\xi = (x^1 - \overline{x}, \ldots, x^k - \overline{x}) \in (\mathbb{R}^p)^k$ identified with $M(p, k; \mathbb{R})$. Since $J$ is differentiable, to show that $J$ is an embedding it suffices to show that $J$ and its derivative are both one to one. If $J([x]_R) = J([x']_R)$, from (4.11) the Euclidean distances between corresponding landmarks of the scaled configurations are equal

$$\|u^i - u^j\| = \|u'^i - u'^j\|, \quad \forall i, j = 1, \ldots, k.$$ (4.12)

Moreover since $\sum_{i=1}^k u^i = 0$, by the fundamental theorem of Euclidean geometry, there is a matrix $T \in O(p)$ such that $u'^i = Tu^i, \forall i = 1, \ldots, k$. If we set $B = \frac{\|\xi\|}{\|\xi_1\|}T$, $b = \overline{x}' - BT\overline{x}$, it follows that $x'^i = Bx^i + b, \forall i = 1, \ldots, k$ with $B^TB = cI_p$, and from (4.4), we see that $[x]_R = [x']_R$. Thus $J$ is one to one. The proof of the injectivity of the derivative of $J$ follows with a similar argument.

**THEOREM 4.2** The range of the Schoenberg embedding of $R\Sigma^k_{p,0}$ is the subset $M_{k,p}$ of $k \times k$ positive semidefinite symmetric matrices $A$ with $\text{rank}(A) = p$, $A1_k = 0$, $\text{Tr}A = 1$, where $1_k$ is the $k \times 1$ column vector $(1 \ldots 1)^T$.

The proof follows as a straightforward application of Theorem 4.1.

4.2.2 Schoenberg Embedding of the $SR\Sigma^k_{p,0}$

Let $S(k, \mathbb{R})$ denote the set of all $k \times k$ real symmetric matrices. The Schoenberg embedding of the size-and-reflection-shape manifold is $J : SR\Sigma^k_{p,0} \rightarrow S(k, \mathbb{R})$, given by

$$J([\xi]_{RS}) = \xi^T \xi$$

(4.13)

where $\xi = (x^1 - \overline{x}, \ldots, x^k - \overline{x}) \in (\mathbb{R}^p)^k$ identified with $M(p, k; \mathbb{R})$. Note that the above expression is used to derive formulas for the extrinsic parameters and their estimators. Since $J$ is differentiable, to show that $J$ is an embedding it suffices for one to show that $J$ and its derivative are both one to one.
THEOREM 4.3 The range of the Schoenberg embedding of $SR_{p,0}^k$ is the subset $SM_{k,p}$ of $k \times k$ positive semidefinite symmetric matrices $A$ with $\text{rank}(A) = p$, $A1_k = 0$.

PROPOSITION 4.4 Let $M_k$ be the space of $k \times k$ symmetric matrices $A$ with $A1_k = 0$. The map $\phi$ from $M_k$ to $S(k - 1, \mathbb{R})$, given by $\phi(A) = HAHT$ is an isometry. In addition, $Tr(\phi(A)) = Tr(A)$.

The outline of 4.4 proof is given in [11]. As a result of 4.4, an embedding $\psi$ of the size-and-reflection-shape manifold as $\psi: SR_{p,0}^k \to S(k - 1, \mathbb{R})$, given by

$$\psi([\xi]_R) = H\xi^T \xi H^T,$$

(4.14)

From Proposition 4.4 it follows that the Schoenberg embedding and the embedding $\psi$ induce the same distance on $SR_{p,0}^k$.

REMARK 4.5 The range of $\psi$ is the set of $(k - 1) \times (k - 1)$ symmetric matrices of rank $p$. Note that for $k = p + 1$, the range is the open convex subset $S_+(k - 1, \mathbb{R}) \subset S(k - 1, \mathbb{R})$ of positive definite symmetric matrices and the induced distance on $SR_{p,0}^k$ is an Euclidean distance.

REMARK 4.6 Let $O(k)$ act on $SR_{p,0}^k$ as $([\xi]_R, A) \to [\xi A]_R, A \in O(k)$. Then the embedding (4.13) is $O(k)$-equivariant. This action is not free. But in view of Proposition 4.4, the Schoenberg embedding can be “tightened” to an $O(k - 1)$-equivariant embedding in $S(k - 1, \mathbb{R})$.

4.3 Extrinsic Means and their Estimators

Recall from Bhattacharyya and Patrangenaru (2003) [3] that the extrinsic mean $\mu_{J,E}(Q)$ of a nonfocal probability measure $Q$ on a manifold $M$ w.r.t. an embedding $J: M \to \mathbb{R}^N$, when it exists, is the point from which the expected squared (induced Euclidean) distance under $Q$ is minimum. It is given by $\mu_{J,E}(Q) = J^{-1}(P_J(\mu))$, where $\mu$ is the usual mean of $J(Q)$ as a probability measure on $\mathbb{R}^N$ and $P_J$ is its projection on $J(M)$ (Bhattacharyya and Patrangenaru (2003), Proposition 3.1 [3]). When the embedding $J$ is given, and the projection $P_J(\mu)$ is unique, one often identifies $\mu_{J,E}$ with its image $P_J(\mu)$, and refer to the later as the extrinsic mean. The term “nonfocal $Q$” means that the projection (minimizer of the distance from $\mu$ to
points in \( J(M) \) is unique. Often the extrinsic mean will be denoted by \( \mu_E(Q) \), or simply \( \mu_E \), when \( J \) and \( Q \) are fixed in a particular context.

Assume \((X_1, \ldots, X_n)\) are i.i.d. \( M \)-valued random objects whose common probability measure is \( Q \), and let \( \bar{X}_E := \mu_{J,E}(\hat{Q}_n) = \mu_E(\hat{Q}_n) \) be the extrinsic sample mean. Here \( \hat{Q}_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{X_j} \) is the empirical distribution. In this section, we present a sketch of the extrinsic mean reflection shape and the extrinsic mean size-and-reflection shape associated with the the Schoenberg emebdding as it was presented in \[11\].

### 4.3.1 Extrinsic Mean Reflection Shape and Extrinsic Mean Size-and-Reflection Shape

Consider a random \( k \)-ad in general position \( X \), centered as \( X_0 = (X^1 - \bar{X}, \ldots, X^k - \bar{X}) \in (\mathbb{R}^p)^k \simeq M(p, k; \mathbb{R}) \), and then scaled to \( U \):

\[
U = X_0/\|X_0\|. \tag{4.15}
\]

Set

\[
C = E(X_0^T X_0), \quad B = E(U^T U). \tag{4.16}
\]

Clearly, the \( Tr(B) = 1, \ B1_k = 0, \ B \geq 0 \) and \( C1_k = 0, \ C \geq 0 \).

The extrinsic mean size-and-reflection-shape of \([X]_{RS}\) exists if \( Tr(C - \xi^T \xi) \) has a unique solution \( \xi \in M(p, k; \mathbb{R}) \) up to an orthogonal transformation, with

\[
\xi_{1k} = 0, \ \text{rank}(\xi) = p. \tag{4.17}
\]

Recall that we remove translation by centering the \( k \)-ad \( x = (x^1, \ldots, x^k) \) to

\[
\xi = (\xi^1, \ldots, \xi^k) \tag{4.18}
\]

\[
\xi^j = x^j - \bar{x}, \forall j = 1, \ldots, k.
\]

That is the same as saying that given \( C \), \( \xi \) is a classical solution in \( \mathbb{R}^p \) to the MDS problem, as given in Mardia et al.,1979 ( p. 408)\[78\] in terms of the first largest \( p \) eigenvalues of \( C \). Assume the eigenvalues of \( C \) in their decreasing order are \( \lambda_1 \geq \cdots \geq \lambda_k \). The classical solution of the MDS problem is unique ( up to an
orthogonal transformation) if $\lambda_p > \lambda_{p+1}$ and $\xi^T$ can be taken as the matrix

$$V = (v_1 v_2 \ldots v_p),$$

(4.19)

whose columns are orthogonal eigenvectors of $C$ corresponding to the largest eigenvalues $\lambda_1 \geq \cdots \geq \lambda_p$ of $C$, with

$$v_j^T v_j = \lambda_j, \forall j = 1, \ldots, p.$$  

(4.20)

Since the eigenvectors $v_1, \ldots, v_p$ are linearly independent, $\text{rk} \xi = p$. If $v$ is an eigenvector of $C$ for the eigenvalue $\lambda > 0$, since $C 1_k = 0$ it follows that $v 1_k = 0$. Therefore the classical solution $\xi$ derived from the eigenvectors (4.19) satisfies (4.17). In conclusion, 4.7

**THEOREM 4.7** Assume $C = \sum_{i=1}^k \lambda_i e_i e_i^T$ is the spectral decomposition of $C = E(X_0^T X_0)$, then the extrinsic mean $\mu_E$ size-and-reflection-shape exists if and only if $\lambda_p > \lambda_{p+1}$ and if this is the case, $\mu_E = [\xi]_{RS}$ where $\xi^T$ can be taken as the matrix (4.19) satisfying (4.20).

From theorem 4.7 it follows that given $k$-ads in general position in $\mathbb{R}^p$, $\{x_1, \ldots, x_n\}$, $x_j = (x_j^1, \ldots, x_j^k)$, $j = 1, \ldots, n$, their extrinsic sample mean size-and-reflection-shape is $[\bar{x}]_E = [\hat{\xi}]_{RS}$, where $\hat{\xi}$ is the classical solution in $\mathbb{R}^p$ to the MDS problem for the matrix

$$\hat{C} = \frac{1}{n} \sum_{j=1}^n \xi_j^T \xi_j.$$  

(4.21)

Here $\xi_j$ is the matrix obtained from $x_j$ after centering, assuming $\hat{\lambda}_p > \hat{\lambda}_{p+1}$. Here $\hat{\lambda}_1 \geq \cdots \geq \hat{\lambda}_k$ are the eigenvalues of $\hat{C}$. Indeed, the configuration of the sample mean is given by the eigenvectors corresponding to the $p$ largest eigenvalues of $\hat{C}$.

**REMARK 4.8** From Remark 4.5, in the case $k = p + 1$, the projection $P_\psi$ is the identity map, therefore any distribution $Q$ is $\psi$-nonfocal and $\psi(\mu_E, \psi)$ is the mean $\mu$ of $\psi(Q)$.

The formula for the extrinsic mean reflection shape of $[X]_R$, has been very recently found by A. Bhattacharya (2009) [80]. This mean exists if

$$\text{Tr}(B - u^T u)^2$$  

(4.22)

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has a unique minimizer \( u \in M(p,k; \mathbb{R}) \) up to an orthogonal transformation, satisfying the constraints
\[
ul_k = 0, \text{Tr}(u^T u) = 1.
\] (4.23)

**THEOREM 4.9** (Abhishek Bhattacharya [80] and Bandulasiri et al. [11]) Assume \( B = \sum_{i=1}^{k} \lambda_i e_i e_i^T \) is the spectral decomposition of \( B \), then the extrinsic mean reflection shape exists if and only if \( \lambda_p > \lambda_{p+1} \). If this is the case, then \( u^T \) can be taken as the matrix
\[
V = (v_1 v_2 \ldots v_p),
\] (4.24)
whose columns are orthogonal eigenvectors of \( B \) corresponding to the largest eigenvalues \( \lambda_1 \geq \cdots \geq \lambda_p \) of \( B \), with
\[
(a) \ v_j^T 1_k = 0 \quad \text{and} \quad (b) \ v_j^T v_j = \lambda_j + \frac{1}{p}(\lambda_{p+1} + \cdots + \lambda_k), \forall j = 1, \ldots, p.
\] (4.25)

4.4 Asymptotic Distribution of Extrinsic Sample Mean Size-and-Reflection Shapes

A comprehensive derivation of the asymptotic distribution of Schoenberg extrinsic sample mean size-and-reflection-shapes is given in A. Bhattacharya (2009) [80]. Here we simply state their results.

Assume that \( Y_1, \ldots, Y_n \) are independent identically distributed random reflection objects from a \( \psi \)-nonfocal probability distribution \( Q \) on \( SR_{\Sigma} \), with \( \lambda_p > \lambda_{p+1} \) and let \( \bar{\psi} \bar{\sigma}(\eta) \) be the mean of \( \psi(Q) \) and \( \Sigma \) be the covariance matrix of \( \psi(Q) \) with respect to the orthobasis \( \tilde{e} \) defined as the following
\[
\Sigma = \begin{pmatrix}
\Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\
\Sigma_{12}^T & \Sigma_{22} & \Sigma_{23} \\
\Sigma_{13}^T & \Sigma_{23}^T & \Sigma_{33}
\end{pmatrix}.
\] (4.26)

If we change the coordinates in \( \mathbb{R}^{k-1} \) by selecting an orthobasis \( \bar{e} \), the eigenvectors \( \bar{e}_1, \ldots, \bar{e}_{k-1} \) of \( \mu \), in such a coordinate system, the mean is a diagonal matrix \( \Lambda \) and
the matrix $\Sigma_\mu = D \Sigma D^T$, where

$$D = \begin{pmatrix}
  I_{p(p-1)/2} & 0 & 0 \\
  0 & \Delta_{p(k-p-1)} & 0 \\
  0 & 0 & 0 \\
\end{pmatrix} \quad (4.27)$$

and

$$\Delta_{p(k-p-1)} = \begin{pmatrix}
  \lambda_1 & \ldots & 0 \\
  \ldots & \ldots & \ldots \\
  0 & \ldots & \lambda_p \\
\end{pmatrix} \quad (4.28)$$

defined in Bhattacharya and Patrangenaru (2005)[71] is

$$\Sigma_\mu = \begin{pmatrix}
  \Sigma_{11} & \Sigma_{12} \Delta & 0 \\
  \Delta \Sigma_{12}^T & \Delta \Sigma_{22} \Delta & 0 \\
  0 & 0 & 0 \\
\end{pmatrix}. \quad (4.29)$$

The extrinsic covariance matrix $\Sigma_E$ defined in Bhattacharya and Patrangenaru (2005), with respect to the basis $d^{-1}(e(\Lambda))$ with $e(\Lambda)$ is

$$\Sigma_E = \begin{pmatrix}
  \Sigma_{11} & \Sigma_{12} \Delta \\
  \Delta \Sigma_{12}^T & \Delta \Sigma_{22} \Delta \\
\end{pmatrix}. \quad (4.30)$$

Let $\tilde{W}$ be the vectorized form of a matrix $W \in S(k-1, \mathbb{R})$ with respect to the basis $\tilde{V}$. Assume $\tan \tilde{W}$ denote the component of $\tilde{W}$ tangent to $\tilde{N}_{p,k}$ at $\tilde{\psi}(\mu)$. 

**THEOREM 4.10** (a) The random vector $n^{1/2} \tan(\tilde{\psi}(Y_E) - \tilde{\psi}(\mu_E))$ converges weakly to a random vector having a $N(0, \Sigma_E)$ distribution, where $\Sigma_E$ is given in (4.30).

(b) If $\Sigma_E$ is nonsingular, then $n \tan(\tilde{\psi}(Y_E) - \tilde{\psi}(\mu_E))^T \Sigma_E^{-1} \tan(\tilde{\psi}(Y_E) - \tilde{\psi}(\mu_E))^T$ converges weakly to a $\chi^2_{(k-1)\frac{p(p+1)}{2}}$ distribution.

From Theorem 4.10 the following result follows:

**COROLLARY 4.11** Let $G$ be a normally distributed matrix in $S(k-1, \mathbb{R})$, weak limit of $n^{1/2}(\tilde{Y} - \mu)$. Assume the spectral decomposition of $\mu$ is $\mu = V \Lambda V^T$. Set $G^V = V^T G V =$
\( \tilde{g}^V_{jl} \) and \( \tilde{G}^V = (\tilde{g}^V_{jl}) \) be determined by

\[
\tilde{g}^V_{jl} = \begin{cases} 
  g^V_{jl} & 1 \leq j \leq l \leq p \\
  \frac{\lambda_j}{\lambda_j - \lambda_l} g^V_{jl} & 1 \leq j \leq p < l \leq p - 1 \\
  0 & p < j \leq l \leq k - 1
\end{cases}
\] (4.31)

Then \( n^{-\frac{1}{2}}(\psi(\bar{Y}_E) - \psi(\mu_E)) \) converges in distribution to the normally distributed random matrix \( VG^VVT \).

From Theorem 4.10 it follows that the extrinsic mean size-and-reflection-shape can be easily estimated using non-pivotal bootstrap. Assume \( \{x_1, \ldots, x_n\} \) is a random sample of configurations \( x_j = (x^1_j, \ldots, x^k_j), j = 1, \ldots, n \). One can resample at random and with repetition \( N \) times from this sample, where \( N \) is a reasonably large number, say \( N \geq 500 \). For each such resample \( x^*_1, \ldots, x^*_n \), compute the extrinsic sample mean \( \bar{x}_{RS}^* \). Then use a local parametrization of \( SR\Sigma_{p,0}^k \) and find \((1 - \alpha)100\%\) Bonferroni simultaneous confidence intervals for the corresponding \( kp - \frac{k(p+1)}{2} \) local coordinates. Similarly, a non-pivotal bootstrap can be used to estimate extrinsic mean reflection shapes.

Two sample tests for extrinsic means on can be be derived from the general theory for two sample tests for extrinsic means on manifolds recently developed by A. Bhattacharya (2008)[81].

**REMARK 4.12** For extrinsic mean size-and-reflection-shapes, \( k = p + 1 \), from Remark 4.5, it follows that the space \( SR\Sigma_{p,0}^{p+1} \) is isometric to a convex open subset of an Euclidean space of dimension \( \frac{p(p+1)}{2} \) and, in view of Remark 4.8 this isometry carries the extrinsic means to ordinary means in this Euclidean space, and inference for means on \( SR\Sigma_{p,0}^{p+1} \) follows from multivariate analysis. In particular, for \( k = p + 1 \), at level \( \alpha \), a nonpivotal bootstrap test for the matched paired hypothesis \( H_0 : \mu_{1,E} = \mu_{2,E} \) can be obtained as follows. Given matched pair samples \( [x_{1,i}]_{RS}, [x_{2,i}]_{RS}, i = 1, \ldots, n \), consider 100\( (1 - \alpha) \) simultaneous confidence intervals for the mean difference matrix \( \psi(\mu_{1,E}) - \psi(\mu_{2,E}) \) obtained from the bootstrap distribution of \( \overline{\psi([x_{1}]_{RS}) - \psi([x_{2}]_{RS})} \), and reject \( H_0 \) if at least one of these intervals does not contain 0.
4.5 Additional Remarks

Rather than using the Schoenberg embedding like A. Bandulasiri et. al. [11], Dryden et. al. (2008)[82] defined a one-to-one function from the space $\Sigma_{p,+}^{k}$ of reflection shapes of k-ads of points that are not all the same, to $S(k − 1, \mathbb{R})$, whose range $N_{p,k}$ is the set of positive semidefinite symmetric matrices $B$ with $\text{rank}(B) \leq p, Tr(A) = 1$. Their embedding $\psi$ is given by

$$\psi([x]_R) = HJ([x]_R)H^T,$$

where $H$ is the $(k − 1) \times k$ Helmert sub-matrix (see Dryden and Mardia (1998), p. 34)[68].

**REMARK 4.13** Note that while the condition in Dryden et al. (2008)[82] for a random reflection shape to be nonfocal is same as A. Bandulasiri et. al. [11], their mean MDS - reflection shape is not the extrinsic mean reflection shape (extrinsic mean under the embedding $\psi$ stated in Theorem 4.9). Indeed their scaling of the eigenvectors $v_1, \ldots, v_p$ corresponding to the $p$ largest eigenvalues of $B$ is given by

$$v_j^T v_j = \frac{\lambda_j}{\lambda_1 + \cdots + \lambda_p}, \forall j = 1, \ldots, p,$$

and the the MDS mean reflection shape is $\mu_{MDS} = [u]_R$. 
CHAPTER 5

INTRINSIC MEANS ON RIEMANNIAN HOMOGENEOUS SPACES

5.1 Intrinsic Means on Hadamard Manifolds

DEFINITION 5.1 Assume \((\Omega, A, P)\) is a probability space, \((\mathcal{M}, g)\) is a complete Riemannian manifold, and \(\mathcal{B}_\mathcal{M}\) is the Borel \(\sigma\)-field generated by open subsets of \(\mathcal{M}\). A random object (r.o.) is a mapping \(X : \Omega \rightarrow \mathcal{M}\), that is \((A, \mathcal{B}_\mathcal{M})\)-measurable.

The probability measure \(Q = P_X\), associated with the r.o. \(X\) is given by \(Q(B) = P(X^{-1}(B)), \forall B \in \mathcal{B}_\mathcal{M}\). Let \(\rho = d_g\) be the geodesic distance on \(\mathcal{M}\). Patrangenaru (1998)[75] considered the function \(F_X : \mathcal{M} \rightarrow \mathbb{R}\), given by

\[
F_X(a) = E\rho^2(X, a) = \int_\mathcal{M} \rho^2(x, a)Q(dx), \forall y \in \mathcal{M}.
\]

Assuming \(F_X(a)\) is finite for some \(a \in \mathcal{M}\), Patrangenaru(2001)[9] called the minimum of \(MF_X\) on \(\mathcal{M}\) to be total intrinsic variance \(t\Sigma_g\) of \(X\) with respect to the metric \(g\). The set of all minimizers of \(F_X\) was called by Patrangenaru (1998)[75] intrinsic mean set of \(X\) with respect to the metric \(g\). When \(F_X\) has a unique minimizer, that minimizer was called the intrinsic mean of \(X\) and is denoted \(\mu_{I,g}\), or simply \(\mu_I\), and noted that \(\mu_{I,g}\) exists if \(\mathcal{M}\) is connected and simply connected, and \((\mathcal{M}, g)\) is complete and has non positive curvature (Patrangenaru(1998))[75]. Such a Riemannian manifold is called a Hadamard manifold.

THEOREM 5.2 The space \((\text{Sym}^+(p), g_F)\) a Hadamard manifold, with the Riemannian exponential mapping at \(I_p\), given by \(\exp_{I_p}(w) = \exp(w), \forall w \in T_{I_p}\text{Sym}^+(p) = \text{Sym}(p)\), where \(\exp(w) = \sum_{r=0}^{\infty} \frac{1}{r!}w^r\).
For a proof of the first claim, from Proposition 2.22, the group $T^+(p, \mathbb{R})$ is a non-commutative simply transitive group of isometries of $\text{Sym}^+(p)$. Moreover, the mapping $t \to h(t) = tt^T = \alpha(t, I_p)$, endows $T^+(p, \mathbb{R})$ with the left invariant Riemannian structure $h^*g_F$, thus $(T^+(p, \mathbb{R}), h^*g_F)$ is a Riemannian homogeneous space in itself. Therefore, $\forall t \in T^+(p, \mathbb{R}), t \to t^{-1}$ is an isometry of $(T^+(p, \mathbb{R}), h^*g_F)$, which is same as saying that for any geodesic $\gamma$, with $\gamma(0) = I_p$, the geodesic reflection on $(T^+(p, \mathbb{R}), h^*g_F)$, at the point $I_p$, given by $\gamma(s) \to \gamma(-s)$, is an isometry of $(T^+(p, \mathbb{R}), h^*g_F)$. Thus $(T^+(p, \mathbb{R}), h^*g_F)$ is a symmetric space in the sense of Cartan at the point $I_p$. This turns the space $(T^+(p, \mathbb{R}), h^*g_F)$, into a globally symmetric space in the sense of Cartan, given that for each point $t \in T^+(p, \mathbb{R})$, the left translation $L_t(u) = tu$ is also an isometry of $(T^+(p, \mathbb{R}), h^*g_F)$, that takes geodesics to geodesics, thus forcing the geodesic reflection at the point $t \in T^+(p, \mathbb{R})$ to be as well an isometry of $(T^+(p, \mathbb{R}), h^*g_F)$. This is a noncompact symmetric space, since $(T^+(p, \mathbb{R})$ is an open subset in a linear space of matrices. From Theorem 3.1. in Helgason(1962) [26] stating that all the sectional curvatures of a symmetric space of noncompact type are non-positive and since an isometry between two Riemannian manifolds preserves the sectional curvature at corresponding points, and $h : (T^+(p, \mathbb{R}), h^*g_F) \to (\text{Sym}^+(p), g_F)$ is an isometry, it follows that $(\text{Sym}^+(p), g_F)$ a Hadamard manifold.

For the second claim, note that a geodesic $\gamma$ on a Lie group $\mathcal{G}$ with a left invariant Riemannian metric, with $\gamma(0) = 1_\mathcal{G}$ is a one parameter subgroup (Helgason, p.94) of $\mathcal{G}$ and it is elementary to show that the one parameter subgroups of $GL(p, \mathbb{R})$ are given by $\gamma(s) = \exp sv$, $\forall s \in \mathbb{R}$, for some $v \in M(p, \mathbb{R})$. In particular, since $\mathcal{G} = T^+(p, \mathbb{R})$ is a Lie subgroup of $GL(p, \mathbb{R})$, the one parameter subgroups of $T^+(p, \mathbb{R})$ are of the form $s \to \exp(sv)$, $\forall s \in \mathbb{R}$, for some $v \in \mathfrak{g}$, where $\mathfrak{g}$ is the set of lower triangular matrices, Lie algebra of $\mathcal{G}$, therefore the geodesics $\gamma_v$ of $(T^+(p, \mathbb{R}), h^*g_F)$, with $\gamma(0) = I_p$, are given by $\gamma(s) = \exp(sv)$, $\forall s \in \mathbb{R}$, for some $v \in \mathfrak{g}$. Finally, since $h$ is an isometry, with $h(I_p) = I_p$, if $s \to \lambda(s)$ is a geodesic on $(\text{Sym}^+(p), g_F)$, with $\lambda(0) = I_p$, then $\gamma(s) = h^{-1}(\lambda(s))$ is a geodesic on $(T^+(p, \mathbb{R}), h^*g_F)$, with $\gamma(0) = I_p$. Thus $\gamma(s) = \exp(sv)$, $\forall s \in \mathbb{R}$, for some $v \in \mathfrak{g}$, and $\lambda(s) = h(\gamma(s)) = \gamma(s)\gamma(s)^T = \exp(sv)(\exp(sv))^T = \exp(sw)$, where $w \in \text{Sym}(p)$.

Given that two distinct points $x, y$ on a Hadamard manifold $(\mathcal{M}, g)$, they can be joined by a unique geodesic

$$
\gamma_{x,y} : \mathbb{R} \to \mathcal{M}, \gamma_{x,y}(0) = x, \gamma_{x,y}(d_g(x, y)) = y, \quad (5.2)
$$

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and an explicit formula of the geodesic $\gamma_{x,y}$ in equation (5.2) is very useful for computing the geodesic distance $d_g(x,y)$. We are in particular interested in this formula $d_{g_F}(x,y)$, given $x,y \in \text{Sym}^+(p)$, which is a direct consequence of theorem 5.2.

**COROLLARY 5.3** The geodesic distance $d_{g_F}(x,y)$ on $\text{Sym}^+(p)$ is given by

$$d_{g_F}^2(y,x) = \text{Tr} \left( \left( \log(c(x)^{-1}x(c(x)^{-1})^T) \right)^2 \right),$$

where $\log : \text{Sym}^+(p) \to \text{Sym}(p)$ is the inverse of the matrix exponential map $\exp : \text{Sym}(p) \to \text{Sym}^+(p)$.

For a proof, recall that $\exp_x : T_x\text{Sym}^+(p) \to \text{Sym}^+(p)$ is the Riemannian exponential at the point $x \in \text{Sym}^+(p)$, $\exp_x(w) = \gamma_{x,w}(1)$, where $\gamma_{x,w}$ it the geodesic on $\text{Sym}^+(p)$ with $\gamma_{x,w}(0) = x, \frac{d}{ds} \gamma_{x,w}(0) = w \in \text{Sym}(p)$. Here, for any point $x \in \text{Sym}^+(p)$, we identify $(x,w) \in T_x\text{Sym}^+(p) = \{x\} \times \text{Sym}(p)$ with $w \in \text{Sym}(p)$. Note that if $x \in \text{Sym}^+(p)$ the then $\alpha_{c(x)} : \text{Sym}^+(p) \to \text{Sym}^+(p)$, is an isometry taking geodesics centered at $I_p$ to geodesics centered at $x$ and since $\alpha_{c(x)}(z) = c(x)zc(x)^T, d_z\alpha_{c(x)}(v) = c(x)vc(x)^T, \forall v \in \text{Sym}(p)$. In addition, it follows that

$$\exp_x(c(x)vc(x)^T) = \gamma_{x,c(x)vc(x)^T}(1) = \gamma_{x,d_{I_p}\alpha_{c(x)}(v)}(1) =$$

$$= \gamma_{x,d_{I_p}\alpha_{c(x)}(v)}(1) = \gamma_{\alpha_{c(x)}(I_p),d_{I_p}\alpha_{c(x)}(v)}(1) =$$

$$= \alpha_{c(x)}(\gamma_{I_p,v}(1)) = c(x)\exp_{I_p}(v)c(x)^T,$$

and from Theorem 5.2 and equation (5.4) we get

$$\exp_x(c(x)vc(x)^T) = c(x)\exp(v)c(x)^T.$$  \hfill (5.5)

Given $w \in T_x\text{Sym}^+(p) \simeq \text{Sym}(p)$, if we solve for $v$ the equation $c(x)vc(x)^T = w$, we get $v = c(x)^{-1}w(c(x)^{-1})^T$, therefore, from equation (5.5) we see that

$$\exp_x(w) = c(x)\exp(c(x)^{-1}w(c(x)^{-1})^T)c(x)^T.$$  \hfill (5.6)

Assume $y = \exp_x(w)$, then $d_{g_F}^2(y,x) = g_{F_x}(w,w)$. One can easily show that

$$g_{F_x}(w,w) = \text{Tr}((x^{-1}w)^2).$$  \hfill (5.7)
We solve equation $y = \text{Exp}(w)$ for $w$ using equation (5.6), to obtain

$$w = c(x) \log(c(x)^{-1}y(c(x)^{-1})^T)c(x)^T,$$  \hfill (5.8)

where log is the inverse of the mapping $\exp : \text{Sym}(p) \rightarrow \text{Sym}^+(p)$, so that log$(\Sigma)$ can be obtained as follows: assume $\Sigma = AA^T$, where $\Lambda$ is the diagonal matrix $\text{diag}(\lambda_1, \ldots, \lambda_p)$, $\lambda_j > 0, \forall j = 1, \ldots, p$, and $A \in O(p)$, then

$$\log(\Sigma) = A \text{diag}(\ln(\lambda_1), \ldots, \ln(\lambda_p))A^T.$$

(5.9)

If we plug $w = w(x, y)$ into the expression on the right hand side of equation (5.7), we get $d^2_{g_F}(x, y) = \text{Tr}((x^{-1}c(x)\log(c(x)^{-1}y(c(x)^{-1})^T)c(x)^T)^2)$. Given that $x = c(x)c(x)^T$, we have $x^{-1} = (c(x)^T)^{-1}c(x)^{-1}$, therefore

$$d^2_{g_F}(y, x) = \text{Tr}((c(x)^T)^{-1}\log(c(x)^{-1}y(c(x)^{-1})^T)c(x)^T)^2), \hfill (5.10)$$

or in a more compact form

$$d^2_{g_F}(y, x) = \text{Tr}((\log(c(x)^{-1}y(c(x)^{-1})^T))^2). \hfill (5.11)$$

The intrinsic mean of a random object $X$ on $\text{Sym}^+(p)$ with respect to the generalized Frobenius distance $g_F$, minimizer of the function $\mathcal{F} : \text{Sym}^+(p) \rightarrow [0, \infty)$, will be called generalized Frobenius mean and is labeled $\mu_F$, and the intrinsic covariance matrix will be called generalized Frobenius covariance matrix will be labeled $\Sigma_F$. As a consequence of equation (5.11) we obtain the following

**COROLLARY 5.4** (a) The generalized Frobenius mean of $X$ is given by

$$\mu_F = \arg\min_{y \in \text{Sym}^+(p)}E[\text{Tr}((\log(c(X)^{-1}y(c(X)^{-1})^T))^2)].$$

(5.12)

The generalized Frobenius mean of a sample $x_1, \ldots, x_n$ of points on $\text{Sym}^+(p)$ is given by

$$\mu_F = \arg\min_{y \in \text{Sym}^+(p)}\sum_{i=1}^n(\text{Tr}((\log(c(x_i)^{-1}y(c(x_i)^{-1})^T))^2)). \hfill (5.13)$$

In order to compute the generalized Frobenius sample mean in (5.13), one may use fast algorithms such as the ones given in Groisser(2004)[73].
CHAPTER 6

3D SIZE-AND-REFLECTION SHAPE ANALYSIS OF THE HUMAN SKULL

The ability to gaze into the human body is an essential diagnostic tool in medicine and is one of the key issues in health care. Medical imaging consists of various techniques and processes used to create images of the human body or parts and function thereof for clinical and medical science purposes. One important reason for using various medical imaging procedures is to reveal, diagnose, and examine diseases in the human body and maybe in some cases, in an animal body. Another important reason for using medical imaging procedures is to study the normal anatomy and physiology of the human body.

In medical imaging, or in bioinformatics shape analysis or size-and-shape based classification is often used to identify mean differences between two populations of planar or volumetric images of the same type of anatomical scenes. Shape changes due to a disease, or caused by aging, as size or volume changes have been widely used.

Here we give a comprehensive application of Chapter 4. For surgery planning, a more natural approach is to take into account the size as well when analyzing the CT scan data. In this context, one performs a nonparametric analysis on the 3D data retrieved from CT scans of healthy young adults, on the on size-and-reflection shape space $SR_3$ of $k$-ads in general position in 3D. As mentioned previously, this work is apart of larger project on planning reconstructive surgery in severe skull injuries. In this work, most of our focus includes preprocessing and post-processing steps of CT images.
6.1 Data Acquisition

The images were taken using a computed assisted tomography or (CAT or CT) scan device. This was done for twenty-eight individuals. A computed assisted tomography scan uses X-rays to make detailed pictures of structures inside of the body. A CAT scan is used to study all parts of the human body, such as the chest, belly, pelvis, or an arm or leg. CAT scan can also take pictures of the body organs, such as the bladder, liver, lungs, pancreas, intestines, kidneys, and heart. In addition, CAT scan can study spinal cord, blood vessels, and the bones. For this reason, specialized software was designed for retrieval of volumetric data from the CAT X-rays.

One CAT scan in our data set consists in about 100+ X-rays (or CT images) of the head above the mandible per individual, for most of the twenty-eight individuals. In Chapter 2 Figure 2.1, we provided an example of a CAT scan for one individual in our data set. Figure 6.2 displays four CT images resulting from a CAT scan of one individual in our data set. Figure 6.1 displays a pictorial flow chart for how CT images such as those in Figure 6.2 and Figure 2.1 are acquired. In addition, Figure 6.1 also provide a windowing scale for the intensity values for a given CT image.

![Flow charts in CT image acquisition](image)

Figure 6.1: Flow charts in CT image acquisition

Image quality is an important issues in medical imaging because these images are an essential diagnostic tool in medicine.

The characteristics of image quality of interest in medical imaging are the following:

- Contrast Sensitivity (very high for CT)
Figure 6.2: CT images of the head for one individual at different elevations

- Blurring and visibility of Detail
- Visual Noise
- Artifacts/Distortion
- Spatial (Tomographic slice or volume views).

Figure 6.3 provides a pictorial example of characteristics of image quality.

Figure 6.3: Characteristics of Medical Image Quality
6.2 Object Extraction

Numerous of methods and systems have been developed for object extraction (thresholding and segmentation) from 2D for 3D display [39, 40, 41, 42]. Visualization of 3D biomedical volume data (images) has traditionally been divided into two different techniques: surface rendering [43, 44, 45, 46, 47, 48] and volume rendering [49, 50, 51, 52, 53, 48, 54]. Both techniques produce a visualization of selected structures in 3D volume data (images), but one should note that the methods involved in these techniques are quite different, and each has its advantages and disadvantages. Selection between these two approaches is often based on the particular nature of the biomedical image data, the application to which the visualization is being applied, the desired result of the visualization, and computational resources. Surface rendering techniques requires the extraction of contours (the edge of the object, in our case the skull) at the 2D slice level and then a tiling algorithm is applied that places surface patches (or tiles) at each contour point and with hidden surface removal and shading, the surface is rendered visible. The advantage of this technique lies in the relatively small amount of contour data, resulting in a fast 3D rendering (reconstruction) speeds. The disadvantages may vary depending on object extraction (or segmentation) software or algorithms. Ideally, one would like to extract all objects of interest from 3D volume data (images) very fast and with high levels of accuracy. In other words, the extracted object should be a good representation of the original object inside of the image. Here we explored various thresholding and segmentation methods in order to extract the bone structure from the CT slices and then perform 3D reconstruction of the virtual skull from these bone extractions.

6.2.1 Thresholding

First we explored thresholding the CT images and then performing 3D reconstruction. We produced threshold images from the original CT images by applying a threshold $T$, based on the gray value. Each pixel of the CT image, $I$, is then classified into two classes: $B$ (bone) and $\overline{B}$ (non-bone). For a pixel $x$ with gray level $I(x)$,

$$
    x \in \begin{cases} 
    B, & \text{if } I(x) \geq T \\
    \overline{B}, & \text{Otherwise}
    \end{cases}
$$

(6.1)
In order to find the threshold value $T$, we grouped the CT images corresponding to each individual into 11 groups, $g$. The groups, $g$, were formed based on the visual inspection, anatomical regions, and the intensity level of material present in a CT image. Here, we make the assumption that the intensity level of each CT image within one of the 11 regions of the human skull would be roughly constant for all CT slices in the same region.

For example, for any CT images within one of the 11 regions, $g$, of the human skull, denoted by $I_{gi}$ where $i = 1, 2, \ldots, 40$, if we randomly select 40 CT images from one of the $g$ groups, we obtain our threshold value $T$ for a group by doing the following:

- We take several horizontal slices across the images $I_{gi}$ (see figure 6.4), within the range of the bone structure, we observed that values below about 100 can be regarded as non-bone $\bar{B}$.

- Next, we find the median intensity value, $Iv$, for each of the images $I_{gi}$, ($med_{gi} = median(Iv(I_{gi}))$).

- Finally, we set $T = median(med_{gi})$, Thus we can regarded non-bone $\bar{B}$ as all values less than $T$.

This method described above thresholds all CT images within one of the 11 regions the same way (see figure 6.5).

![Figure 6.4: Semi-Thresholding Example](image)

Figure 6.4 displays threshold images using the method that we describe in 6.2.1.

Figure 6.6 displays threshold images using the method that we describe in 6.2.1.
Despite all of our hard efforts, the 3D reconstruction based on the 2D thresholding method presented above applied to the CT images did not yield very good visual 3D reconstructions. Select results are displayed in figure 6.7. As a result, we turned our attention towards segmentation methods.

6.2.2 Segmentation: Minimizing the Geodesic Active Contour

Segmentation is a well studied area and it is usually formulated as the minimization of a cost/energy function subjected to some constraints. Segmenting 3D image volumes slice by slice using image processing techniques is a lengthy process and requires a post-processing step to connect the sequence of 2D contours into a continuous surface (3D reconstruction). Caselles et al. (1997)\cite{83} introduced the Geodesic Active Contour (GAC) model, as an enhanced version of the snake model of (Kass et. al., 1988)\cite{84}. The GAC model is defined as the following variation problem:

$$\min_C \{ E_{GAC}[C] \},$$

where $E_{GAC}[C] = \int_0^{|C|} g(\|\nabla I(C(s))\|)dl$. \hspace{1cm} (6.2)
In (6.2) $|C|$ is the Euclidean length of the curve $C$ and $dl$ the Euclidean element of arc. The edge detection function, $g \in (0, 1]$ in Eq. 6.2 has the following meanings: values close to 0 are at strong edges in the image $I$ whereas, values close to 1 are not at edges in the image $I$. $|\nabla I|$ acts as an edge detector. In particular, $\nabla I$ is the gradient of the gray level along the curve $C(s)$ [83]. A (local) minimal distance path between given points is a geodesic curve. To show this (Caselles et al., 1997) [83] used the classical Maupertuis Principle from dynamical systems [83] which essentially explains when an energy minimization problem is equivalent to finding a geodesic curve in a Riemannian space (Dubrovin et al., 1984). Typically, to find the global optimal solution of equation 6.2, graph based approaches are commonly used which rely on partitioning of a graph that is built based on the image $I$. Unfortunately, such approaches could lead to major systematic discretization error problems. Talbot and Appleton(2006)[85] presented an approach that mini-

Figure 6.6: Threshold images of the head for one individual at different elevations
mizes the \( GAC \) energy using continuous maximal flows. The amazing gain from their approach is that it does not suffer from any discretization errors. Bresson et al. (2005)\cite{86} produced a different approach, which uses the \textit{weighted Total Variation}. The \textit{weighted Total Variation} or simply \textit{weighted TV} is defined as

\[
TV_g(u) = \int_{\Omega} g(x) |\nabla u| d\Omega. \tag{6.3}
\]

\( TV_g(u) \) is the weighted gradient of \( u \). The active contour \( C \) is a level-set of a function \( u : [0, a] \times [0, b] \rightarrow \mathbb{R} \). In other words, \( u \) is an implicit representation of the active curve \( C \), since \( C \) coincides with the set of points \( u = \text{constant} \). Bresson et al. showed that under certain conditions, namely if \( u \) is a characteristic function \( 1_C \), then Eq. (6.3) is equivalent to \( E_{GAC} \) in (6.2). The details is provided in Bresson et al. (2005)\cite{86}. In order to find the geodesic curve, the corresponding steepest-descent flow of Eq. (6.3) is computed. If we allowed \( u \) to vary continuously between \([0, 1]\), then Eq. (6.3) becomes a convex function, meaning that one can compute the global minimizer of it. Unger et al. (2008)\cite{55} proposed the following variational image segmentation model:

\[
\min_{u \in [0,1]} \{ E_{Seg} \}
\]

where \( E_{Seg} = \int_{\Omega} g(x)|\nabla u| d\Omega + \int_{\Omega} \lambda(x)|u - f| d\Omega. \tag{6.4} \)

Here the first term of the energy is the \textit{weighted TV} of \( u \) as defined in Eq. (6.3), which minimizes the \( GAC \) energy. The second term is used to incorporate constraints into the energy function. The variable \( f \in [0, 1] \) is provided by the use and it indicates foreground (\( f = 1 \)) and background (\( f = 0 \)) seed regions. The spatially varying parameter \( \lambda(x) \) is responsible for the interpretation of the information con-
obtained in \( f \). Figure 6.8 displays ten 3D reconstruction based on Unger et. al. (2008) [55] method summarized above. Appendix C displays twenty 3D reconstruction based on Unger et. al. (2008) [55] method summarized above.

Figure 6.8: Select 3D Reconstruction Results via segmentation

6.3 Landmark Based Skull Data Analysis

Once we obtained the 3D reconstruction of the virtual skull from the bone extractions, we proceed to perform landmark based analyses based on the Schoenberg embedding. For the purpose of one analysis we were interested in \( k = 9 \) and \( k = 17 \) matched landmarks around the eyes. Appendix A displays coordinates
of the $k = 9$ landmarks for each of the twenty 3D reconstructed skulls. Similarly, Appendix B displays coordinates of the $k = 17$ landmarks for each of the twenty 3D reconstructed skulls. The landmarks were registered on the reconstructed 3D virtual skulls. Typically, landmarks are chosen by an expert in the field in which the imaging data originated to represent points of interest, i.e. anatomical features, or some mathematical property, such as curvature.

Here we consider nonparametric statistical analysis size-and-reflection shape data using landmarks in which each observation $x = (x_1, \ldots, x^9)$ and $x = (x_1, \ldots, x^{17})$ consists of 9 points and 17 points in $\mathbb{R}^3$ (See Figure 6.9 and Figure 6.10).

Figure 6.9: Groups of landmarks displayed on the 20 virtual skulls, $k = 9$

We remove translation by centering the $k$-ads $x = (x_1, \ldots, x^9)$ and $x = (x_1, \ldots, x^{17})$
Figure 6.10: Two groups of landmarks around the eye: Right image has $k = 9$ and the left image has $k = 17$

$$\xi = (\xi^1, \ldots, \xi^9) \text{ and } \xi = (\xi^1, \ldots, \xi^{17})$$

$$\xi^j = x^j - \overline{x}, \forall \ j = 1, \ldots, 9 \text{ and } j = 1, \ldots, 17.$$ 

The set of all centered $k$-ads lie in a vector subspace $L^3_{9} \in (\mathbb{R}^3)^9$ and $L^3_{17} \in (\mathbb{R}^3)^{17}$, respectively. The dimension of the manifolds $SR\Sigma^9_{3,0}$ and $SR\Sigma^{17}_{3,0}$ is given by formula $pk - \frac{p(p+1)}{2}$.

In order to estimate the 3D size-and-reflection shape for the selected group of landmarks and pseudo-landmarks, we compute the Schoenberg sample means (see Chapter 4). That is, we used 500 bootstrap resamples based on the original 20 skull configurations ($k = 9$ and $k = 17$), represented by the 3 by k matrices (where k was the number of landmarks selected in the analysis). For the purpose of one analysis we were interested in $k = 9$ and $k = 17$ landmarks around the eyes region. Conveniently selected representations for these mean size-and-reflection shapes yield the bootstrap mean size-and-reflection shape configuration given in Figure 6.11 and Figure 6.12, respectively.

In addition, for $k = 9$ we provide a 90% simultaneous confidence limits for the 3D mean size-and-reflection shape Configuration are given in Tables 6.1 and 6.2.
Figure 6.11: Bootstrap distribution for the Schoenberg sample mean configurations based on 500 resamples

Table 6.1: 90\% Lower Confidence Limit for the Bootstrap Distribution of the 3D Sample Mean Size-and-Reflection Shape Configuration, $k = 9$

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6.4 Discussion and Future Work

For future work, we propose to estimate the Schoenberg mean size-and-reflection shape of k-ads in general position in $\mathbb{R}^3$ for the human skull for a larger selection of landmarks (for some $k$: $21 \leq k \leq 61$). In addition, we also propose to find the bootstrap distribution of the Schoenberg sample means 3D size-and-reflection shape for a select group of anatomic landmarks and pseudo-landmarks for 500 bootstrap resamples of the original 20 skulls represented by the 3 by $k$ configu-
Figure 6.12: Bootstrap distribution for the Schoenberg sample mean configurations based on 500 resamples

Table 6.2: 90% Upper Confidence Limit for the Bootstrap Distribution of the 3D Sample Mean Size-and-Reflection Shape Configuration, $k = 9$

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Table 6.3: 90% Lower Confidence Limit for the Bootstrap Distribution of the 3D Sample Mean Size-and-Reflection Shape Configuration, $k = 17$

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Table 6.4: 90% Upper Confidence Limit for the Bootstrap Distribution of the 3D Sample Mean Size-and-Reflection Shape Configuration, $k = 17$

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...rations, for some $k$ ($21 <= k <= 61$). Next, we will report a confidence region for the bootstrap distribution of the Schoenberg mean configurations. In addition, we will propose methodology to estimate landmark selection experimenter error (LSEE) between two anatomic landmark based configurations. Our proposed estimate is based on the bootstrap distribution of the Schoenberg sample means 3D size-and-reflection shape for two anatomic landmark based configuration for 500 and 2000 bootstrap resamples of the original 20 skulls.

Finding geometric correspondence amongst multiple individuals is a difficult and an important part of landmark based 2D or 3D size-and-reflection shape analysis. Here we propose to find geometric correspondence amongst the 3D extracted
bone structures by defining the boundary of the bone structure in each CT image slice to be two planar contours. The outer and inner contour represents the outside and inside of the bone structure respectively. Next we will apply Ellingson et al. (2010) [87] idea here to find the proportional corresponding sites (pseudo landmarks) for various CT slices above the eye.

Nonparametric function estimation of the skull thickness will be performed at all of sites (pseudo landmarks) above the eye. Our long term goal is to fuse all statistical information gained with neurological information in planning for certain surgical procedures. We will propose the use of resulting bone thickness maps, to predict the thickness of the skull at the location of an injury site for planning reconstructive surgery in severe skull injuries.
CHAPTER 7

TWO-SAMPLE TESTS FOR INTRINSIC MEANS ON HOMOGENEOUS HADAMARD MANIFOLDS

7.1 Two-Sample Test

Assume the Hadamard homogeneous space \( M, d_g \) admits a simply transitive group of isometries \( G \). Under these assumption, a two sample problem for intrinsic means on \( M \), can be transferred to a one sample problem on the Lie group \( G \), as follows. Fact that the left action \( \alpha : G \times M \to M \) is simply transitive, means that in the isotropy group \( G_x \) is a trivial subgroup consisting in the identity element \( 1_G \). More generally, for any fixed object \( x \in M \), the mapping \( \alpha^x : G \to M \) is bijective, therefore the two sample mean hypothesis testing for linear data \( H_0 : \mu_1 = \mu_2 + \delta \) vs. \( H_1 : \mu_1 \neq \mu_2 + \delta \), that is formulated in our more general setting for intrinsic means as follows:

\[
(1) \quad H_0 : \mu_{1,g} = \alpha(\delta, \mu_{2,g}) \\
\text{versus} \\
H_1 : \mu_{1,g} \neq \alpha(\delta, \mu_{2,g}) 
\]

(7.1)

can be translated to a hypothesis testing problem on the Lie group \( G \), as follows:

\[
(1) \quad H_0 : (\alpha^{\mu_1,g})^{-1}(\mu_{2,g}) = \delta, \\
\text{versus} \\
H_1 : (\alpha^{\mu_1,g})^{-1}(\mu_{2,g}) \neq \delta 
\]

(7.2)
Let $H : \mathcal{M}^2 \to \mathcal{G}$, defined by

$$H(x_1, x_2) = (\alpha^{x_1})^{-1}(x_2). \quad (7.3)$$

**THEOREM 7.1** Assume $X_{a,j_n}, j_n = 1, \ldots, n_a$ are identically independent distributed random objects (i.i.d.r.o.’s) from the independent probability measures $Q_a$, $a = 1, 2$ with finite intrinsic moments of order $s$, $s \leq 4$ on the $m$ dimensional Hadamard manifold $\mathcal{M}$ on which the Lie group $\mathcal{G}$ acts simply transitively. Let $n = n_1 + n_2$ and assume $\lim_{n \to \infty} \frac{n_1}{n} \to p \in (0, 1)$. Let $\exp : \mathfrak{g} \to \mathcal{G}$ and $L_\delta$ be respectively, the Lie group exponential map, and the left translation by $\delta \in \mathcal{G}$. Then under $H_0$,

i. The sequence of random vectors

$$\sqrt{n}((L_\delta \circ \exp)^{-1}(H(\bar{X}_{n_1,g}, \bar{X}_{n_2,g}))) \quad (7.4)$$

converges weakly to $\mathcal{N}_m(0_m, \Sigma_g)$, for some covariance matrix $\Sigma_g$ that depends linearly on the intrinsic covariance matrices $\Sigma_{a,g}$ of $Q_a$, $a = 1, 2$.

ii. If (i.) holds and $\Sigma_g$ is positive definite, then the sequence

$$n((L_\delta \circ \exp)^{-1}(H(\bar{X}_{n_1,g}, \bar{X}_{n_2,g})))^T \Sigma_g^{-1}((L_\delta \circ \exp)^{-1}(H(\bar{X}_{n_1,g}, \bar{X}_{n_2,g}))) \quad (7.5)$$

converges weakly to $\chi^2_m$ distribution.

By the multivariable inverse function theorem, the mapping $H : \mathcal{M} \times \mathcal{M} \to \mathcal{H}$ is continuous. Given that, according to Bhattacharya and Patrangenaru(2003)[3], for $a = 1, 2$, the intrinsic sample mean $\bar{X}_{n_a,g}$ is a consistent estimator of $\mu_{a,g}$, for $a = 1, 2$, by the continuity theorem (Billingsley(1995)[88], p.334) a consistent estimator for $(\alpha^{\mu_1,g})^{-1}(\mu_{2,g})$ is $H(\bar{X}_{n_1,g}, \bar{X}_{n_2,g})$. From Bhattacharya and Patrangenaru(2005)[71], under the null hypothesis, for $a = 1, 2$, the asymptotic distribution of $\sqrt{n_a}Exp_{\mu_{a,g}}^{-1}(\bar{X}_{n_1,g})$ is multivariate normal $\mathcal{N}_m(0_m, \Sigma_a)$, where $\Sigma_a$ is the intrinsic covariance matrix of $Q_a$ (the intrinsic covariance matrix is the covariance matrix in the CLT for intrinsic means in Bhattacharya and Patrangenaru (2005)[71]). Since $\lim_{n \to \infty} \frac{m_1}{n} \to p \in (0, 1)$, from Cramér’s delta method it follows that asymptotically $\sqrt{n}((L_\delta \circ \exp)^{-1}(H(\bar{X}_{n_1,g}, \bar{X}_{n_2,g})))$ has $\mathcal{N}_m(0_m, \Sigma)$, where $\Sigma$ depends on the intrinsic covariance matrices of the two populations $X_{a,1}$, $a = 1, 2$ as described in the statement of a. in the theorem. Part b. is an immediate consequence of part a. and a weak
continuity argument (Billingsley(1995)[88], p.334).

**COROLLARY 7.2** For \( a = 1, 2 \), assume \( x_{a,j_a}, j_a = 1, \ldots, n_a \), are random samples from independent populations on the \( m \) dimensional Hadamard manifold \( \mathcal{M} \) on which the Lie group \( \mathcal{G} \) acts simply transitively. Let \( n = n_1 + n_2 \), and assume \( \lim_{n \to \infty} \frac{n_1}{n} \to p \in (0, 1) \). Assume \( \Sigma_g \) is positive definite and \( \hat{\Sigma}_g \) is a consistent estimator for \( \Sigma_g \). The asymptotic \( p \)-value for the hypothesis testing problem 7.1 is given by \( p = P(T \geq T_\delta^2) \) where

\[
T_\delta^2 = n((L_\delta \circ \exp)^{-1}(H(\bar{x}_{n_1,g}, \bar{x}_{n_2,g}))^T(\hat{\Sigma}_g)^{-1}((L_\delta \circ \exp)^{-1}(H(\bar{x}_{n_1,g}, \bar{x}_{n_2,g}))).
\]

(7.6)

Often in practice, the sample sizes are too small to obtain reliable statistical results using parametric models. If the distributions are unknown and the samples are small, an alternative approach is for one to use nonparametric bootstrap. If \( n_a \leq m, \forall a = 1, 2 \), the estimator \( \hat{\Sigma}_g \) in Corollary 7.2 does not have an inverse, and pivotal nonparametric bootstrap methodology can not be applied. In this case, one may use nonpivotal bootstrap for the two sample problem \( H_0 \), following Bhattacharya and Ghosh(1978)[89], Babu and Singh(1984)[90], Hall and Hart(1990)[91], Fisher et. al.(1996)[92], Hall(1997)[93] and others.

**THEOREM 7.3** Under the hypotheses of Theorem 7.1i., assume in addition, that for \( a = 1, 2 \) the distribution of \( \text{Exp}^{-1}_{\mu_{A,1}} X_{a,1} \) has an absolutely continuous component, and finite moments of sufficiently high order. Then the joint distribution of

\[
V = \sqrt{n}(\exp)^{-1}(H(\bar{X}_{n_1,g}, \bar{X}_{n_2,g}))
\]

can be approximated by the bootstrap joint distribution of

\[
V^* = \sqrt{n}(\exp)^{-1}(H(\bar{X}^*_{n_1,g}, \bar{X}^*_{n_2,g}))
\]

(7.7)

with an error \( O_p(n^{-\frac{1}{2}}) \), where, for \( a = 1, 2 \), \( \bar{X}^*_{n_a,g} \) are the intrinsic means of the bootstrap resamples \( X^{*}_{a,j_a}, j_a = 1, \ldots, n_a \), given \( X_{a,j_a}, j_a = 1, \ldots, n_a \).
7.2 Tests for Equality of Generalized Frobenius Means via Cholesky Decompositions

In this section, we apply the results developed previously in Section 7.1 of Chapter 7 to the case of distributions the Hadamard manifold \((\mathcal{M}, g) = (\text{Sym}^+(p), g_F)\) with the simple transitive group action of \(G = T^+(p, \mathbb{R})\) given in Proposition 2.22.

From equation (5.12), the generalized Frobenius sample mean of a sample \(x_1, \ldots, x_n\) of matrices on \(\text{Sym}^+(p)\) is given by

\[
\mu_F = \arg\min_{y \in \text{Sym}^+(p)} \frac{1}{n} \sum_{i=1}^{n} [\text{Tr}((\log(c(x_i)^{-1}y(c(x_i)^{-1}T))^2)],
\]

(7.8)

where \(\forall i = 1, \ldots, n, x_i = c(x_i)c(x_i)^T\) is the Cholesky decomposition of \(x_i\).

**REMARK 7.4** The Cholesky decomposition was used for DTI also by Wang et. al. (2004)[94], although they did not use the generalized Frobenius distance in their paper.

From Section 7.1, we see that given two independent populations on \(\text{Sym}^+(p)\), with generalized Frobenius means \(\mu_{a,F}, a = 1, 2\), the testing problem

\[
H_0 : \mu_{1,F} = \delta \mu_{2,F} \delta^T
\]

versus

\[
H_1 : \mu_{1,F} \neq \delta \mu_{2,F} \delta^T
\]

(7.9)

is equivalent to testing on \(T^+(p, \mathbb{R})\)

\[
H_0 : c(\mu_{1,F})c(\mu_{2,F})^{-1} = \delta
\]

versus

\[
H_1 : c(\mu_{1,F})c(\mu_{2,F})^{-1} \neq \delta.
\]

(7.10)

For testing, suppose for \(a = 1, 2\) we are given the i.i.d.r.o.’s \(X_{a,1}, \ldots, X_{a,n_a} \in \text{Sym}^+(p)\) with the total sample size \(n = n_1 + n_2\). For \(a = 1, 2\), the corresponding sample generalized Frobenius means are \(\bar{X}_{a,F}\). Note that, in our case, the matrix valued function \(H\) in equation (7.3) is given by

\[
H(x_1, x_2) = c(x_1)c(x_2)^{-1}.
\]

(7.11)
A consistent estimator of $H(\mu_{1,F}, \mu_{2,F})$ is $T = H(\bar{X}_{1,F}, \bar{X}_{2,F})$, and Theorem 7.1 becomes

**THEOREM 7.5** Assume $X_{a,j_a}, j_a = 1, \ldots, n_a$ are i.i.d.r.o.’s from the independent probability measures $Q_a, a = 1, 2$ generalized Frobenius moments of order $s, s \leq 4$ on the $\text{Sym}^+(p)$. Let $n = n_1 + n_2$ and assume $\lim_{n \to \infty} \frac{n}{n} \to p \in (0, 1)$. Let $\log : T^+(p, \mathbb{R}) \to T(p, \mathbb{R})$ be the inverse of $\exp$ given in a neighborhood of $I_p$ by

$$\log(I_p + v) = v - \frac{1}{2}v^2 + \cdots + \frac{(-1)^r + 1}{r!}v^r + \ldots, \forall v \in T(p, \mathbb{R}), Tr(vv^T) < 1. \quad (7.12)$$

Then under $H_0$,

i. The sequence of random vectors

$$\sqrt{n}((\delta^{-1}(H(\bar{X}_{1,F}, \bar{X}_{2,F})))) \quad (7.13)$$

converges weakly to $\mathcal{N}_{\frac{p(p+1)}{2}}(0, \Sigma_F)$, for some covariance matrix $\Sigma_F$ that depends linearly on the generalized Frobenius covariance matrices $\Sigma_{a,F}$ of $Q_a, a = 1, 2$.

ii. If (i.) holds and $\Sigma_F$ is positive definite, then the sequence

$$n((\delta^{-1}(H(\bar{X}_{1,F}, \bar{X}_{2,F}))))^T \Sigma^{-1}_F (\delta^{-1}(H(\bar{X}_{1,F}, \bar{X}_{2,F})))) \quad (7.14)$$

converges weakly to $\chi^2_{\frac{p(p+1)}{2}}$ distribution.

Theorem 7.3, becomes in our case

**THEOREM 7.6** Under the hypotheses of Theorem 7.5i., assume in addition, that for $a = 1, 2$ the distribution of $\text{Exp}^{-1}_{\mu_{F,a}} X_{a,1}$ has an absolutely continuous component, and finite moments of sufficiently high order. Then the joint distribution of

$$V_\delta = \sqrt{n}(\log(\delta^{-1}(H(\bar{X}_{1,F}, \bar{X}_{2,F}))))$$

can be approximated by the bootstrap joint distribution of

$$V_\delta^* = \sqrt{n}(\log(\delta^{-1}(H(\bar{X}_{1,F}^*, \bar{X}_{2,F}^*)))) \quad (7.15)$$

with a coverage error $O_p(n^{-\frac{1}{2}})$, where, for $a = 1, 2$, $\bar{X}_{a,F}^*$ are the sample generalized Frobenius means of the bootstrap resamples $X_{a,j_a}, j_a = 1, \ldots, n_a$, from $X_{a,j_a}, j_a = 1, \ldots, n_a$. 63
Each bootstrap resample in Theorem 7.6 of the total sample, is obtained by sampling with replacement from the samples of sizes \( n_1, n_2 \) from \( Q_1, \) respectively \( Q_2. \) The matrix valued bootstrap statistics \((7.15)\) are recomputed \( N \) times. To insure accuracy, one usually takes \( N \) to be at least 5,000. The bootstrap values of \( V \) and \( \hat{T} \) are given by:

\[
\hat{T}^* = \hat{T}_1^* \hat{T}_2^{-1*} \quad \text{and} \quad V_\delta^* = \log(\delta^{-1}\hat{T}^*), \quad \text{where} \quad (7.16)
\]

\[
\hat{T}_a^* = c(\bar{X}_{a,F}), a = 1, 2.
\]

We construct a \((1 - \alpha)\) bootstrap confidence region \( C_\alpha^* \) for \( \tau = c(\mu_{1,F}c(\mu_{2,F})^{-1} \) and a \((1 - \alpha)\) bootstrap confidence region \( R(\delta)_\alpha^* \) and for \( \log(\delta^{-1}\tau) \) respectively, based on the bootstrap distributions of \( \hat{T}^* \) and of \( V_\delta^*. \)

**Corollary 7.7**

i. We fail to reject \( H_0 \) in \((7.10)\), at level \( \alpha \), if \( \delta \in C_\alpha^* \), with an error \( O_p(n^{-\frac{1}{2}}) \).

ii. We fail to reject \( H_0 \) in \((7.10)\), at level \( \alpha \), if \( 0_p \in R(\delta)_\alpha^* \), with an error \( O_p(n^{-\frac{1}{2}}) \).

**Remark 7.8**

It is known that there are many choices for bootstrap confidence regions (see eg. Fisher et. al.(1996)[92]). Computationally, it is often convenient to use simultaneous confidence intervals, even if we the coverage is a bit enlarged. In this case, we reject \( H_0 \) at level \( \alpha \) if \( \delta \notin C_\alpha^* \) or equivalently \( 0_p \notin R(\delta)_\alpha^* \), that is the same as saying that we reject \( H_0 \) if the Bonferroni 100\((1 - \alpha)\)% simultaneous bootstrap confidence intervals for the \( \tau_{ij}, i \leq j \) or for \( \log(\delta^{-1}\tau)_{ij}, i \leq j \). Such Bonferroni simultaneous bootstrap confidence intervals are formed by cutting off the lower and the upper \( 100\left(\frac{\alpha}{2m}\right) \) \% of the bootstrap distributions of \( \hat{T}_{ij}^* \) and \( V_{ij}^* \), respectively. Here \( m = \frac{p(p+1)}{2} \).
CHAPTER 8
DIFFUSION TENSOR IMAGING
APPLICATION

8.1 Dyslexia Detection via DTI & Motivation

In this Chapter, we apply our new methodology, presented in Section 7.2 of Chapter 7, to a concrete DTI example, using a dataset kindly provided by A. Schwartzman. The data was collected from two groups of children, a group of 6 children with normal reading abilities and a group of 6 children with a diagnosis of dyslexia. Twelve spatially registered diffusion MRIs (DT images) was obtained from the two groups of children, respectively. The National Institute of Neurological Disorders and Stroke (NINDS) gives the following definition for dyslexia: “Dyslexia is a brain-based type of learning disability that specifically impairs a person’s ability to read” [95]. Individuals with dyslexia more often than not find themselves reading at levels significantly lower than expected despite having normal intelligence [95].

The dyslexia disorder varies from person to person. According to the NINDS, some common characteristics among people with dyslexia are difficulty with spelling, phonological processing (the manipulation of sounds), and/or rapid visual-verbal responding. In adults, dyslexia usually occurs after a brain injury or in the context of dementia. Dyslexia can also be inherited in some families, and recent studies have identified a number of genes that may predispose an individual to developing dyslexia [95]. For those with dyslexia, the prognosis is mixed. The disability affects such a wide range of people and produces such different symptoms and varying degrees of severity that predictions are hard to make. The prognosis is generally good, however, for individuals whose dyslexia is identified early, who
have supportive family and friends and a strong self-image, and who are involved in a proper remediation program. In Figure 8.1, we display DTI slices including a given voxel recorded in a control subject and a dyslexia subject (first subject of Table 8.1 and Table 8.2).

Commonly in DTI group studies, a typical statistical problem is to find regions of the brain whose anatomical characteristics differ between two groups of subjects. Typically, the analysis consists of registering the DT images to a common template so that each voxel corresponds to the same anatomical structure in all the images, and then applying two-sample tests at each voxel.

Here, we present the analysis of a single voxel at the intersection of the corpus callosum and corona radiata in the frontal left hemisphere that was found in Schwartzman et al. (2008) [22] to exhibit the strongest difference between the two groups, based on a parametric data analysis. Table 8.1 shows the data at this voxel for the clinically normal (control) subjects, whereas Table 8.1 data at this voxel for the dyslexia subjects. The $d_{ij}$ in Table 8.1 and Table 8.2 are the entries of the DT on and above the diagonal (the below-diagonal entries are the same since the DTs are symmetric).

For this analysis, our primary goal is to demonstrate that our nonparametric two-sample testing procedure, presented in Section 7.2, is able to detect a significant difference between of the generalized Frobenius means of the clinically normal and dyslexia groups without increasing the dimensionality in the process. Namely, we are interested in detecting, on average, from Diffusion Tensor Images (DTI), dyslexia in young children compared to their clinically normal peers, with-
Table 8.1: DTI data for the control group

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{11}$</td>
<td>0.8847</td>
<td>0.6516</td>
<td>0.4768</td>
<td>0.6396</td>
<td>0.5684</td>
<td>0.6519</td>
</tr>
<tr>
<td>$d_{22}$</td>
<td>0.9510</td>
<td>0.9037</td>
<td>1.1563</td>
<td>0.9032</td>
<td>1.0677</td>
<td>0.9804</td>
</tr>
<tr>
<td>$d_{33}$</td>
<td>0.8491</td>
<td>0.7838</td>
<td>0.6799</td>
<td>0.8265</td>
<td>0.7918</td>
<td>0.7922</td>
</tr>
<tr>
<td>$d_{12}$</td>
<td>0.0448</td>
<td>-0.0392</td>
<td>0.0217</td>
<td>0.0229</td>
<td>-0.0427</td>
<td>0.0269</td>
</tr>
<tr>
<td>$d_{13}$</td>
<td>-0.1168</td>
<td>-0.0631</td>
<td>-0.0091</td>
<td>-0.1961</td>
<td>-0.0879</td>
<td>-0.1043</td>
</tr>
<tr>
<td>$d_{23}$</td>
<td>0.0162</td>
<td>-0.0454</td>
<td>-0.1890</td>
<td>-0.1337</td>
<td>-0.1139</td>
<td>-0.0607</td>
</tr>
</tbody>
</table>

Table 8.2: DTI data for the dyslexia group

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{11}$</td>
<td>0.5661</td>
<td>0.6383</td>
<td>0.6418</td>
<td>0.6823</td>
<td>0.6159</td>
<td>0.5643</td>
</tr>
<tr>
<td>$d_{22}$</td>
<td>0.7316</td>
<td>0.8381</td>
<td>0.8776</td>
<td>0.8376</td>
<td>0.7296</td>
<td>0.8940</td>
</tr>
<tr>
<td>$d_{33}$</td>
<td>0.8232</td>
<td>1.0378</td>
<td>1.0137</td>
<td>0.9541</td>
<td>0.9683</td>
<td>0.9605</td>
</tr>
<tr>
<td>$d_{12}$</td>
<td>0.0358</td>
<td>-0.0044</td>
<td>-0.0643</td>
<td>0.0309</td>
<td>-0.0929</td>
<td>-0.0635</td>
</tr>
<tr>
<td>$d_{13}$</td>
<td>-0.2289</td>
<td>-0.2229</td>
<td>-0.1675</td>
<td>-0.2217</td>
<td>-0.1713</td>
<td>-0.1307</td>
</tr>
<tr>
<td>$d_{23}$</td>
<td>-0.1106</td>
<td>-0.0449</td>
<td>-0.0192</td>
<td>-0.0925</td>
<td>-0.0965</td>
<td>-0.1791</td>
</tr>
</tbody>
</table>

out making any distributional assumptions.

8.2 Data Analysis of Diffusion Tensor Images

Given two independent populations with i.i.d. samples of random SPD matrices $X_{1,1}, X_{1,2}, \ldots, X_{1,n_1} \in Sym^+(3)$ from the clinically normal population and $X_{2,1}, X_{2,2}, \ldots, X_{2,n_2} \in Sym^+(3)$ from the dyslexia population with sample sizes of $n_1 = 6$ and $n_2 = 6$ and the total sample size $n = n_1 + n_2 = 12$, where, for $a = 1, 2, X_{a,1} \sim \mu_{F,a}$, the sample generalized Frobenius mean for the clinically normal population and dyslexia population is given by
\[ \bar{x}_{1,F} = \begin{pmatrix} 0.6318 & 0.0046 & -0.0924 \\ 0.0046 & 0.9863 & -0.0873 \\ -0.0924 & -0.0873 & 0.7803 \end{pmatrix} \quad \text{and} \quad \bar{x}_{2,F} = \begin{pmatrix} 0.6146 & -0.0261 & -0.1910 \\ -0.0261 & 0.8118 & -0.0901 \\ -0.1910 & -0.0901 & 0.9537 \end{pmatrix}. \]

The test statistics \( \hat{T} \) and \( V \), previously described in Section 7.2, are given by

\[ \hat{T} = \begin{pmatrix} 0.9862 & 0.0000 & 0.0000 \\ -0.0485 & 0.9067 & 0.0000 \\ -0.1487 & -0.0152 & 1.0781 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} -0.0139 & 0.0000 & 0.0000 \\ -0.0513 & -0.0980 & 0.0000 \\ -0.1446 & -0.0153 & 0.0752 \end{pmatrix}. \]

In addition, let \( \hat{t}_{ij} \) and \( v_{ij} \) correspond to the entries of the test statistics \( \hat{T} \) and \( V \) on and below the diagonal (since the test statistics \( \hat{T} \) and \( V \) are lower triangular matrices).

In order to test hypothesis (7.9) or hypothesis (7.10), for \( \delta = I_3 \), we repeatedly resample observations from the original data and compute the generalized Frobenius sample mean for each respective group. The generalized Frobenius sample means are computed as described in Chapter 5. Tables 8.3 and 8.4 display a five number summary for the bootstrap distribution of the Generalized Frobenius sample means for the clinically normal and dyslexia groups. To generate the summary statistics presented in tables 8.3 and 8.4, we used 10,000 bootstrap resamples. Figure 8.2 and Figure 8.3 displays a visualization of the bootstrap distributions of the Generalized Frobenius sample means.

In addition, for each bootstrap resample, we calculate the Cholesky decomposition of the bootstrap generalized Frobenius sample mean for each respective group and then proceed to calculate the bootstrap distribution of our test statistics.
Table 8.3: Five Number Summary of the bootstrap distribution of the generalized Frobenius sample means for the clinically normal group

<table>
<thead>
<tr>
<th>Quartiles</th>
<th>Generalized Frobenius sample means</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Min</td>
</tr>
<tr>
<td>d₁₁</td>
<td>0.4902</td>
</tr>
<tr>
<td>d₂₂</td>
<td>0.9018</td>
</tr>
<tr>
<td>d₃₃</td>
<td>0.6951</td>
</tr>
<tr>
<td>d₁₂</td>
<td>-0.0422</td>
</tr>
<tr>
<td>d₁₃</td>
<td>-0.1852</td>
</tr>
<tr>
<td>d₂₃</td>
<td>-0.1788</td>
</tr>
</tbody>
</table>

Table 8.4: Five Number Summary of the bootstrap distribution of the generalized Frobenius sample means for the dyslexia group

<table>
<thead>
<tr>
<th>Quartiles</th>
<th>Generalized Frobenius sample means</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Min</td>
</tr>
<tr>
<td>d₁₁</td>
<td>0.5626</td>
</tr>
<tr>
<td>d₂₂</td>
<td>0.7269</td>
</tr>
<tr>
<td>d₃₃</td>
<td>0.8232</td>
</tr>
<tr>
<td>d₁₂</td>
<td>-0.0880</td>
</tr>
<tr>
<td>d₁₃</td>
<td>-0.2289</td>
</tr>
<tr>
<td>d₂₃</td>
<td>-0.1791</td>
</tr>
</tbody>
</table>
Figure 8.2: Bootstrap distribution for the generalized Frobenius sample means for $d_{11}, d_{22}, d_{33}, d_{12}, d_{13},$ and $d_{23}$; clinically normal (red) vs dyslexia (blue)

Figure 8.3: Bootstrap distribution for the generalized Frobenius sample means for $d_{11}, d_{22}, d_{33}, d_{12}, d_{13},$ and $d_{23}$; clinically normal (red) vs dyslexia (blue)

$\hat{T}$ and $V$ as described in (7.16). Figures 8.4 and 8.5 displays a visualization of our non-pivotal bootstrap distribution of our test statistics $\hat{T}$ and $V$.

Under the null hypothesis (7.10), $\delta = I_3$ on $T^+(3, \mathbb{R})$ or $\log(\delta^{-1}) = 0_3$ on the vector space $T(3, \mathbb{R})$; however, after visually examining Figures 8.4 and 8.5, we
Figure 8.4: Bootstrap distribution of our test statistics $\hat{T}$: The images (1 - 3) in the first row corresponds to the diagonal entries of the matrices $\hat{T}^*_{t_{11}, t_{22}, t_{33}}$ and images (4 - 6) in the second row corresponds to the lower triangular off-diagonal entries of the matrices $\hat{T}^*_{t_{21}, t_{31}, t_{32}}$

informally conclude that there is significant difference between the generalized Frobenius means of the clinically normal and dyslexia group, since the $\hat{T}^*_{t_{22}}$ and $\hat{V}^*_{v_{22}}$

Figure 8.5: Bootstrap distribution of our test statistics $V$: The images (1 - 3) in the first row corresponds to the diagonal entries of the matrices $V^*: v_{11}, v_{22}, v_{33}$ and images (4 - 6) in the second row corresponds to the lower off-diagonal entries of the matrices $V^*: v_{21}, v_{31}, v_{32}$

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values does not overlap with $\delta_{22} = 1$, respectively with $0_{3,22} = 0$. Moreover, we also observed that the distributions of $\hat{T}_{33}$, $V_{33}$ and $\hat{T}_{31}$, $V_{31}$ barely touch $\delta_{33} = 1$, $0_{3,33} = 0$ and $\delta_{31} = 0$, $0_{3,31} = 0$.

These results formally confirm at level $\alpha$, that there is significant evidence that the clinically normal and dyslexia children display on average different DTI responses. The result were obtained by performing a $100(1-\alpha)$%-simultaneous bootstrap confidence intervals, as described in Remark (7.8), for $\hat{T}_{ij}$ and $V_{ij}$. Tables 8.5 and 8.6 display the results of the Bonferroni $100(1-\alpha)$%-simultaneous bootstrap confidence intervals for $\hat{T}_{ij}$ and $V_{ij}$ at the following significance level: $\alpha = 0.06, 0.03,$ and $0.006$, where for each marginal we consider a $100(1-\frac{\alpha}{12})$% confidence interval. The significant differences are marked with an asterisk in those tables.
8.3 Discussion and Future Work

To our knowledge, the idea of using nonparametric techniques in Diffusion Tensor Image analysis, was first advanced in the working group of Data Analysis on Sample Spaces with a Manifold Stratification, as part of the 2010-2011 program of Analysis of Object Data at the Statistical and Applied Mathematics Institute in the Research Triangle, North Carolina. The work presented here focuses on two sample tests for data analysis on the regular part of the stratification of the space of positive semi-definite matrices in dimension $p$. It might be useful to expand this work also to cases when the generalized Frobenius means are on the singular strata. Such data analysis for DTI, might be used for example, to detect branching points of brain vessels, regions where the blood is not flowing properly, etc. Another extension, would be in considering that given the consistence of the intrinsic sample means, one should actually look at locally homogeneous models for analyzing DTI data, or CMB data, or equally challenging data sets.
## APPENDIX A

### TABLE OF NINE ANATOMICAL LANDMARKS ON THE 20 SKULLS

<table>
<thead>
<tr>
<th>Landmark No.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>69</td>
<td>84</td>
<td>105</td>
<td>82</td>
<td>111</td>
<td>120</td>
<td>140</td>
<td>153.8</td>
<td>139</td>
</tr>
<tr>
<td>y</td>
<td>55.88</td>
<td>45.06</td>
<td>42</td>
<td>46.12</td>
<td>30.4</td>
<td>43</td>
<td>48.01</td>
<td>57</td>
<td>46.27</td>
</tr>
<tr>
<td>z</td>
<td>44</td>
<td>52</td>
<td>47.06</td>
<td>33</td>
<td>38</td>
<td>46.17</td>
<td>32</td>
<td>43</td>
<td>51</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Landmark No.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>67.14</td>
<td>83</td>
<td>101.4</td>
<td>79</td>
<td>108</td>
<td>115</td>
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## APPENDIX B

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APPENDIX C

3D RECONSTRUCTION OF BONE STRUCTURE VIA SEGMENTATION APPROACH
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